7.1 Nonlinear Systems

Review of Classifications

1. \[ x' = x + ty \]
   \[ y' = 2x + y + \gamma \sin t \]
   Dependent variables: \( x, y \)
   Parameter: \( \gamma \)
   Nonautonomous linear system
   Nonhomogeneous \((\gamma \sin t)\) system

2. \[ u' = 3u + 4\nu \]
   \[ \nu' = -2u + \sin t \]
   Dependent variables: \( u, \nu \)
   Parameters: none
   Nonautonomous linear system
   Nonhomogeneous \((\sin t)\) system

3. \[ x_1' = \kappa x_2 \]
   \[ x_2' = -\sin x_1 \]
   Dependent variables: \( x_1, x_2 \)
   Parameter: \( \kappa \)
   Autonomous nonlinear \((\sin x_1)\) system

4. \[ p' = q \]
   \[ q' = pq - \sin t \]
   Dependent variables: \( p, q \)
   Parameters: none
   Nonautonomous nonlinear \((pq)\) system

5. \[ S' = -rSI \]
   \[ I' = -rSI - \gamma I \]
   \[ R' = \gamma I \]
   Dependent variables: \( R, S, I \)
   Parameters: \( r, \gamma \)
   Autonomous nonlinear \((SI)\) system

Verification Review

6. \[ x' = x \]
   \[ y' = y \]
   Substituting \( x = y = e^t \) into the two differential equations, we get
   \[ e^t = e^t \]
   \[ e^t = e^t \]

7. \[ x' = y \]
   \[ y' = -x \]
   Substituting \( x = \sin t \) and \( y = \sin t \) into the two differential equations, we get
   \[ \cos t = \cos t \]
   \[ -\sin t = -\sin t \]

8, 9. Use direct substitution, as in Problems 6 & 7.
For the phase portraits of Problems 10-13 we focus on the \textit{slope information} contained in the DEs, using
the following general principles:

\begin{itemize}
  \item Setting \( x' = 0 \) gives the \textit{v-nullcline} of vertical slopes
  \item Setting \( y' = 0 \) gives the \textit{v-nullcline} of vertical slopes
  \item The equilibria are located where an \textit{h-nullcline} intersects a \textit{v-nullcline}, i.e., where \( x' = 0 \) and \( y' = 0 \) simultaneously
  \item In the regions between nullclines the DEs tell whether trajectories move left or right (sign of \( x' \)), up or down (sign of \( y' \))
  \item The direction picture that results shows the stability of the equilibria
\end{itemize}

Note: if the trajectories circle around an equilibrium, further argument is necessary to distinguish between a center and a spiral (which could be either stable or unstable).

Note: For computer-drawn trajectories, the Runge-Kutta method will be far more accurate than Euler's method at answering these questions.

Note: Recall from Section 6.5 that an equilibrium with trajectories that head toward it in one direction and others that head away in \textit{another} direction is a \textit{saddle}. Unique trajectories (\textit{separatrices}) head to or from a saddle and \textit{separate} the behaviors.

10. \( x' = y \) \hspace{1cm} \text{v-nullcline:} \hspace{0.2cm} y = 0
    \[ y = x(x-1) \] \hspace{1cm} \text{h-nullclines:} \hspace{0.2cm} x = 0 \text{ and } x = 1

Equilibria: \((0,0),(1,0)\)

Because some direction arrows point away from \((1,0)\) the second equilibrium is unstable; in fact because other direction arrows point toward it, this equilibrium is more precisely a saddle.

Because direction arrows circle around \((0,0)\) the first equilibrium could be either a center or a spiral. We argue that the symmetry of the direction field for positive and negative \( y \) implies trajectories must circle rather than spiral, so the equilibrium is a center point.

The trajectories and vector field confirm all of the above information; see figures. Most trajectories come from the lower right, bend around the equilibria, and leave at the upper right, except those that circle through points between \((0,0)\) and \((1,0)\), the equilibria, the separatrices of the saddle.

\begin{figure}[h]
  \centering
  \includegraphics[width=0.4\textwidth]{nullclinesDirections.png}
  \includegraphics[width=0.4\textwidth]{typicalTrajectories.png}
  \caption{Nullclines and directions \hspace{1cm} Typical trajectories}
\end{figure}
11. \[ x' = xy \] \[ y' = y - 3\cos x \]

v-nullclines: x- and y- axes
h-nullclines: \( y = 3\cos x \)

Equilibria: \( (0, 3), (\pm \frac{\pi}{2}, 0), (\pm \frac{3\pi}{2}, 0), (\pm \frac{5\pi}{2}, 0) \)...

The nullclines/direction figure indicates that the equilibrium at \( (0, 3) \) is unstable, and that those on the \( x \)-axis alternate between saddles and centers (or spirals), starting with saddles at those closest to the origin. To settle the question of centers or spirals, we note that the direction fields are not symmetric across the \( x \)-axis (as in Problem 10), and that the trajectories shown in Figure 7.1.1(b) imply unstable spirals.

The trajectories and vector field confirm all of the above information; see figures. Because none of the equilibria are stable, all trajectories must conclude at infinity somewhere. In the upper half plane they go to the left or right, and in the lower half plane they go to minus infinity along the \( y \)-axis. All solutions originate at \( t = -\infty \) near one of the unstable equilibria.

12. \[ x' = x^2 - y + 2 \] \[ y' = y + 2x \]

v-nullclines: \( y = x^2 + 2 \)

h-nullclines: \( y = -2x \)

Equilibria: none, because h- and v-nullclines do not intersect.

The nullclines/direction figure indicates that trajectories come from the left and move to the right; those with large enough \( y \)-values to cross the parabola will head back to the left and move upward toward the left half of the parabola, otherwise they move toward the right forever. The trajectories and vector field confirm all of the above information; see figures.
13. \[ x' = 1 + y + x^2 \quad v\text{-nullclines: } y = x^2 - 1 \]
\[ y' = y/2 - 3x \quad h\text{-nullclines: } y = 6x \]

Equilibria: approximately at (-0.2, -1.2), (-5.83, -34.98)

The first equilibrium at (-0.2, -1.2) is a spiral because directions are circling around it and the direction field is not symmetric. The trajectories in Fig 7.1.1(d) indicate it is an unstable spiral from which nearby trajectories seem to be attracted to a limit cycle. The trajectories and vector field confirm all of the above information; see figures.

The second equilibrium is off the scale of these pictures, in the lower left. An extended nullcline sketch will show it to be a saddle (Note that the arrows on one nullcline will switch direction when it crosses another.). To confirm this analysis, make an extended vector field with trajectories.
Phase Portraits from Nullclines

For Problems 14-19, note the general procedures listed with Problems 10-13. Note that these procedures when combined with computer pictures of trajectories and vector fields should give redundant information. That is, if ever any of these do not agree, you can know there is an error. Furthermore, it should not matter in what order you apply these procedures. (E.g., if the nullclines are difficult to plot, as in Problem 16, you might start with a computer phase portrait and vector field, then use the slope marks to locate and sketch approximately the nullclines.) Focus on looking for (and checking for) consistency.

14. \( x' = xy \)
\[ y' = y - x^2 + 1 \]

Equilibria: (0,-1), (1,0), (-1,0)

The equilibria are all unstable, as confirmed by the trajectories and vector fields in the figures. Almost all trajectories emanate from near one of the unstable equilibria and then circle around clockwise or counterclockwise, eventually approaching minus infinity along the \( y \)-axis. The exceptions are the equilibria and trajectories that begin on the \( y \)-axis above \( y = -1 \) and go straight up towards +\( \infty \).

15. \( x' = y - \ln |x| \)
\[ y' = x - \ln |y| \]

Equilibria: approximately at (-1.31, 0.27), (-0.57, -0.57), (0.27, 1.31)

The equilibria in the 2nd and 4th quadrants are saddle points (hence unstable); the equilibrium in the 3rd quadrant is an unstable node. The trajectories and vector field confirm all of the above information; see figures.

Questions arise, however. Note that in the computer pictures some trajectories cross the axes while others stop there. Should they cross or not? Technicaly the DEs are not defined when \( x = 0 \) or \( y = 0 \); why should they appear to have the same slopes on either side of an axis? Does symmetry help find answers? Learn to be on the lookout for issues like this that require additional analysis; even if you don't have answers, it is important to list any unresolved questions.
16. \[ x' = y + x(1 - x^2 - y^2) \quad \text{v-nullclines: } y + x(1 - x^2 - y^2) = 0 \]
\[ y' = -x + y(1 - x^2 - y^2) \quad \text{h-nullclines: } -x + y(1 - x^2 - y^2) = 0 \]

Equilibria: (0,0)

These nullclines are not simple curves; they have been drawn with a computer package. They intersect at only one point, the origin, which is an unstable spiral. The phase portrait shows a stable limit cycle lying along the unit circle, which attracts all nonequilibrium trajectories. This is a good example to explain in terms of polar coordinates: when a point is on the unit circle, the DE reduces to the harmonic oscillator. The trajectories and vector field confirm all of the above information; see figures.

17. \[ x' = 1 - x^2 - y^2 \quad \text{v-nullclines: } x^2 + y^2 = 1 \]
\[ y' = x \quad \text{h-nullclines: } y\text{-axis} \]

Equilibria: (0,1), (0,-1)

The equilibrium at (0,-1) is an unstable saddle. The equilibrium at (0,1) is stable, and with the symmetry of the direction field about the y-axis it is a stable center, surrounded by closed periodic orbits. Note: These closed orbits are not limit cycles, because they do not attract or repel nearby trajectories. Note also that these closed orbits are nested within an exceptional orbit emanating from the saddle below. This "saddle connection" (see Hubbard and West, Part II, Ch 9) is an example of how an apparently simple system of DEs can lead to an unusual set of behaviors.

The trajectories and vector field confirm all of the above information; see figures. All trajectories not trapped inside the saddle connection move from right to left, wiggling around it if necessary.
18. \[
\begin{align*}
x' &= y - x^2 + 1 & \text{v-nullclines: } y &= x^2 - 1 \\
y' &= y + x^2 - 1 & \text{h-nullclines: } y &= -x^2 + 1
\end{align*}
\]
Equilibria: \((-1,0), (1,0)\)

The equilibrium at \((-1, 0)\) is an unstable saddle. The equilibrium at \((1,0)\) is an unstable spiral (the direction field is not symmetric about either \(x\) or \(y\) direction, and the direction arrows spiral us outward). The trajectories and vector field confirm all of the above information; see figures.

19. \[
\begin{align*}
x' &= |x| - y - 1 & \text{v-nullclines: } y &= -|x| + 1 \\
y' &= |x| + y - 1 & \text{h-nullclines: } y &= |x| - 1
\end{align*}
\]
Equilibria: \((-1,0), (1,0)\)

The equilibrium at \((-1, 0)\) is an unstable saddle. The equilibrium at \((1,0)\) is an unstable spiral (the direction field is not symmetric about either \(x\) or \(y\) direction, and the direction arrows carry us on an outward spiral.) The trajectories and vector field confirm all of the information; see figures.
Equilibria for Second-Order DEs

For second order DEs (e.g., Problems 20-25) we find a lovely shortcut to determining directions: Our first order system begins with introducing a new variable $y$ for the first derivative, e.g., $x' = y$; this immediately determines that trajectories move

- to the right in the upper half plane,
- to the left in the lower half plane, and
- vertically when they cross the $x$-axis.

\[ x'' + (x^2 - 1)x' + x = 0 \]

(a),(b) Letting $y = x'$ we obtain the first order system

\[
\begin{align*}
  x' &= y \\
  y' &= -x - (x^2 - 1)y
\end{align*}
\]

v-nullclines: $y = 0$ (x-axis)

h-nullclines: $y = x/(1-x^2)$

Equilibrium: $(0,0)$

The horizontal nullcline is a rational function; sketching it is a good review of calculus graphing with asymptotes.

(c) The figures show that the equilibrium at (-1, 0) is an unstable saddle, with trajectories spiraling out to a limit cycle. The solution $x(t) = 0$ of the second-order DE is unstable.

(d) The limit cycle shown in the phase portrait attracts trajectories from outside as well as inside, so it represents a stable periodic solution. This is another van der Pol's equation representing oscillations in certain nonlinear electrical circuits, as in Example 3.
21. \( \theta'' + (g / L) \sin \theta = 0 \)

(a), (b) Letting \( y = \theta' \) we obtain the first order system

\[
\begin{align*}
\theta' &= y \\
y' &= -\left( \frac{g}{L} \right) \sin \theta
\end{align*}
\]

v-nullclines: \( y = 0 \) (\( \theta' \)-axis)

h-nullclines: \( \theta = \pi / n, \ n = 0, \pm 1, \pm 2 \ldots \)

Equilibria: \( (n / \pi, \theta), \ n = 0, \pm 1, \pm 2 \ldots \)

(c), (d) The figures (for \( g = L = 1 \)) show that the equilibrium points

\( (0,0), (\pm 2\pi, 0), (\pm 4\pi, 0) \ldots \)

are center points (hence stable), because trajectories near them circle around and form closed loops (by symmetry of the slope marks about the \( x \)-axis); equilibrium points

\( (\pm \pi, 0), (\pm 3\pi, 0), (\pm 5\pi, 0) \ldots \)

are saddles (hence unstable). The constant solutions \( x(t) = 0, \pm 2\pi, 4\pi, \ldots \) of the second-order DE

are stable. The constant solutions \( x(t) = \pi, 3\pi, 5\pi \ldots \) are unstable.

22. \( x'' - x / (x - 1) = 0 \)

(a), (b) Letting \( y = x' \) we obtain the first order system

\[
\begin{align*}
x' &= y \\
y' &= x / (x - 1)
\end{align*}
\]

v-nullclines: \( y = 0 \) (\( x' \)-axis)

h-nullclines: \( x = 0 \) (\( y' \)-axis)

Equilibria: \( (0,0) \)

(c), (d) The figures show that the origin appears to be a center with periodic solutions moving clockwise around it; by the symmetry of the slope marks about the \( x \)-axis we can argue that the trajectories form closed loops. Hence \( x(t) = 0 \) is a stable solution of the second-order DE.

Note that as trajectories approach \( x = 1 \) (where the DE is not defined), the \( y \)-derivative approaches \( +\infty \) or \( -\infty \); trajectories tend to move straight up (for \( x > 1 \)) or straight down (for \( x < 1 \)). The periodic solutions are not limit cycles because none attract nearby solutions.
23. \[ x'' + (x')^2 + x^2 = 0 \]

(a),(b) Letting \( y = x' \) we obtain the first order system
\[ x' = y \]
\[ y' = -x^2 - y^2 \]

v-nullclines: \( y = 0 \) (x-axis)

h-nullclines: \( x^2 + y^2 = 0 \) (the origin)

Equilibrium: \((0,0)\)

(c),(d) The figures show that the origin is unstable. Although one trajectory heads directly toward the origin, another heads away from it and all others pass it by. Hence \( x(t) = 0 \) is an unstable solution of the second-order DE. There are no periodic solutions.

24. \[ x'' + |x|x' + x = 0 \]

(a),(b) Letting \( y = x' \) we obtain the first order system
\[ x' = y \]
\[ y' = -|x|x' - x \]

v-nullclines: \( y = 0 \) (x-axis)

h-nullclines: \( x + |x|y = 0 \);

i.e., the y-axis as well as \( y = \begin{cases} 1, x < 0 \\ -1, x > 0 \end{cases} \)

Equilibrium: \((0,0)\)

(c),(d) The figures show that trajectories spiral into the origin (note that the slope marks are not symmetric about either axis). Hence the origin is a stable spiral equilibrium point, and \( x(t) = 0 \) is a stable solution of the second-order DE. There are no periodic solutions.
25. \( x'' + ((x')^2 - 1)x' + x = 0 \)

(a), (b) Letting \( y = x' \) we obtain the first order system

\[
\begin{align*}
  x' &= y \\
  y' &= -(x - (y^2 - 1)y)
\end{align*}
\]

v-nullclines: \( y = 0 \) (x-axis)

h-nullclines: \( x + (y^2 - 1)y = 0 \)

Equilibrium: \((0,0)\)

(c), (d) The figures show that trajectories circle around the origin, but we note that the slope marks are not symmetric about either axis, so we expect a spiral. Furthermore, the phase portrait shows a noncircular limit cycle surrounding the origin that attracts solutions from within as well as all solutions that begin outside the cycle. The origin is thus an unstable spiral equilibrium point, and \( x(t) = 0 \) is an unstable solution of the second-order DE.

![Nullclines and directions](image1)

![Typical trajectories](image2)

- Creative Challenge

26. Student Project

- Finding Equations for Trajectories

27. \( x' = y, \ y' = x \)

We write \( \frac{dy}{dx} = \frac{y'}{x'} = \frac{x}{y} \)

Separating variables yields \( ydy = xdx \). Hence,

\[
\frac{1}{2}y^2 = \frac{1}{2}x^2 + c
\]

or \( x^2 - y^2 = 2c \)

This is a family of hyperbolas in the phase plane, which can be seen in the figure. The direction that trajectories follow can be determined by looking at the original system. From equations \( x' = y \) and \( y' = x \) we see that trajectories in the first and third quadrants move away from the origin, but in the second and fourth quadrants, trajectories move towards the origin.
### 28. \[ x' = y, \quad y' = -x \]

We write \[ \frac{dy}{dx} = \frac{y'}{x'} = -\frac{x}{y} \]

Separating variables yields \( ydy = -xdx \). Hence,

\[
\frac{1}{2} y^2 = -\frac{1}{2} x^2 + c
\]

or \( x^2 + y^2 = c \)

This is a family of circles, which we have drawn in the phase plane. The direction that trajectories follow can be determined by looking at the original system. From equations \( x' = y \) and \( y' = -x \) we see that when \( x \) is positive \( y \) decreases and when \( x \) is negative \( y \) increases. Hence, movement is in the clockwise direction.

### 29. \[ x' = y(x^2 + 1), \quad y' = 2xy^2 \]

We write \[ \frac{dy}{dx} = \frac{y'}{x'} = \frac{2xy^2}{y(y^2 + 1)} = \frac{2xy}{x^2 + 1} \]

Separating variables yields

\[
\frac{dy}{y} = \frac{2x}{x^2 + 1} \, dx,
\]

so

\[
\ln |y| = \ln(x^2 + 1) + c
\]

or

\[
|y| = e^c (x^2 + 1)
\]

Hence, \( y = C(x^2 + 1) \),

where \( C \) is an arbitrary constant. This family of parabolas is shown in the figure; the direction of the trajectories is determined by the signs of \( x' \) and \( y' \). Note: For this system, the entire \( x \)-axis is a line of unstable equilibrium points where \( x' = y' = 0 \).
30. \( x' = 1, \ y' = x + y \)

We write \( \frac{dy}{dx} = \frac{y'}{x'} = x + y \),

which we can solve easily as a linear equation, yielding

\[ y = ce^x - x - 1 \]

See figure for this family of curves in the phase plane. We can determine the direction in which solutions move along the trajectories because \( x' > 0 \) means \( x \) is always increasing.

### Nonlinear Systems from Applications

31. \( \dot{x} = f(x,y) = 2xy \)
\( \dot{y} = g(x,y) = y^2 - x^2 - 1 \)

Solving \( \dot{x} = \dot{y} = 0 \), shows the system has two equilibrium points \((0,\pm 1)\). Trajectories move on elliptical paths from an unstable equilibrium at \((0,1)\) to a stable equilibrium at \((0,-1)\).

32. \( \dot{x} = 2xy \)
\( \dot{y} = y^2 - x^2 \)

Solving \( \dot{x} = \dot{y} = 0 \), shows the system has one equilibrium point \((0,0)\). The trajectories leave this point on elliptical paths and return asymptotically back to the origin. Hence the origin is a merger of an unstable equilibrium with a stable equilibrium.
33. \[ \dot{x} = y \]
\[ \dot{y} = -x - \text{sgn}(y) \]

Solving \( x' = y' = 0 \), we find the single equilibrium point \((0,0)\). We draw several trajectories showing that when the trajectory crosses the \( x \)-axis between \(-1 \) and \(+1 \), the trajectory simply stops and turns around, only to turn around again and again, giving rise to a "chattering" motion.

Note: The figure suggests there exist equilibria at \((\pm1, 0)\), but algebra shows they are not, because \( y' = 1 \neq 0 \) at those points.

We can interpret the solution physically as a vibrating spring represented by the single equation
\[ x'' + \text{sgn}(x') + x = 0, \]
where the friction always opposes the direction of motion of the spring with constant magnitude 1. When the displacement \( x \) is small, the friction force is stronger than the spring force \(-x\), with the net result that when \(-1 \leq x \leq 1\), trajectories simply chatter back and forth across the \( x \)-axis.

![Trajectories of Coulomb damping](image1)

![Zoom on chattering effect](image2)

Note: We saw this chattering in a phase portrait only when the approximation crosses the \( x \)-axis, which it does for Euler's method with \( h = 0.5 \). Our plots by Euler at \( h = 0.1 \) or by Runge-Kutta at \( h = 0.5 \) did not show the chattering phenomenon.

**Sequential Solutions**

34. \[ x' = -2x \]
\[ y' = xy^2 \]

We start by solving the first equation, yielding \( x(t) = c_1e^{-2t} \).

We then substitute this into the second equation, yielding
\[ y^{-2} \, dy = c_1e^{-2t} \, dt, \]

which can be integrated to give
\[ -y^{-1} = \frac{1}{2}c_1e^{-2t} + c_2. \]

Hence, the system has the general solution
\[ x(t) = c_1e^{-2t} \]
\[ y(t) = \frac{2}{c_1e^{-2t} + c_3}. \]
Polar Limit Cycles

Phase portraits for Problems 35-38 can be most easily sketched by hand. However, if you wish to make a computer drawing with an xy DE solver, note that

\[ x'(r \cos \theta)' = r' \cos \theta - r \theta' \sin \theta \]
\[ y'(r \sin \theta)' = r' \sin \theta + r \theta' \cos \theta \]

Rewrite the right-hand sides in terms of \( x \) and \( y \) using

\[ \cos \theta = x/r, \quad \sin \theta = y/r, \quad r = \sqrt{x^2 + y^2} \]

and the given expressions for \( r' \) and \( \theta' \).

35. \[ r' = (1 - r)^2 \]
\[ \theta' = 1 \]

Because \( r' = 0 \) when \( r = 0 \) or \( r = 1 \), we note that the origin is an equilibrium and that the constant solution \( r = 1 \) gives a closed trajectory in the \( xy \) phase-plane.

The equation \( \theta' = 1 \) tells us that the trajectories rotate around the origin at a constant angular velocity (1 radian per unit time) in the counterclockwise direction, regardless of \( r \)-value, so the origin is the only equilibrium.

We also see (algebraically) that \( r' > 0 \) for all \( r \neq 1 \), hence \( xy \)-trajectories for \( r < 1 \) very slowly approach the unit circle, but trajectories for \( r > 1 \) very slowly move away from it. Thus \( r = 1 \) is a (semistable) limit cycle, stable from the inside and unstable from the outside.

36. \[ r' = r(a - r) \]
\[ \theta' = 1 \]

Because \( r' = 0 \) when \( r = 0 \) or when \( r = a \), we have an equilibrium at the origin and a closed trajectory at \( r = a \) in the \( xy \) phase-plane.

The equation \( \theta' = 1 \) tells us that trajectories rotate around the origin at a constant angular velocity (1 radian per unit time) in the counterclockwise direction, regardless of \( r \)-value, so the origin is the only equilibrium.

We also note that \( r' > 0 \) for \( 0 < r < a \), and that \( r' < 0 \) for \( r > a \), so \( xy \)-trajectories both inside and outside the circle \( r = a \) approach it asymptotically. Hence \( r = a \) is a stable limit cycle.
37. \[ r' = r(1-r)(2-r) \]
\[ \theta' = 1 \]

The equation \( \theta' = 1 \) tells us that the trajectories rotate around the origin at a constant angular velocity (1 radian per unit time) in the counterclockwise direction.

The equation for \( r \) tells us that \( r' > 0 \) and \( r \) is increasing for \( 0 < r < 1 \) and for \( r > 2 \);
\( r' < 0 \) and \( r \) is decreasing for \( 1 < r < 2 \). See graph for \( dr/dt \) versus \( r \).

Hence \( xy \)-trajectories approach the circle \( r = 1 \) from both inside and outside, and move away from the circle \( r = 2 \) on both the inside and outside, as shown in the \( xy \) phase portrait. The origin is an unstable equilibrium, and there are limit cycles at \( r = 1 \) (stable) and \( r = 2 \) (unstable).

38. \[ r' = r(1-r)(2-r)(3-r)^2 \]
\[ \theta' = 1 \]

By reasoning as in Problem 37, the origin is an unstable equilibrium, and there are limit cycles at \( r = 1 \) (stable), \( r = 2 \) (unstable) and \( r = 3 \) (semistable). See figures.

We have added a \( tx \) graph, because it shows nicely the stability of the limit cycles: those that are stable show periodic cycles in forward time; those that are unstable show periodic cycles in backward time.
Testing Existence and Uniqueness

39. \[ x' = 1 + x = f(x, y) \]
\[ y' = (1 + x)\sqrt{y} = g(x, y) \]

(a) The domain of the system of differential equations is the upper-half \( xy \)-plane; that is, all points \((x, y)\) for which \( y \geq 0 \). Also because \( f \) is continuous everywhere, and \( g \) is continuous for \( y \geq 0 \), there exists a solution passing through each point \((x_0, y_0)\) for which \( y_0 \geq 0 \) (the existence theorem does not tell how far the solution can be extended).

We now compute for partial derivatives
\[
\frac{\partial f}{\partial x} = 1, \quad \frac{\partial f}{\partial y} = 0
\]
\[
\frac{\partial g}{\partial x} = \sqrt{y}, \quad \frac{\partial g}{\partial y} = \frac{1 + x}{2\sqrt{y}}
\]

Which are continuous for \( y > 0 \), and so there exists a unique solution passing through each point \((x_0, y_0)\) for which \( y_0 > 0 \).

(b) The direction field as shown in the figure seems to confirm the analysis so far, and looks suspiciously nonunique along \( y = 0 \) as indeed we have seen in Section 1.5, Example 3.

However, this is definitely not the whole story for the system given. Although the system reduces to \( y' = \sqrt{y} \) when \( t \) is eliminated, and although the extended theorem assures us there will be a unique solution whenever \( y > 0 \), the fact that we have a system of two equations introduces the need to also look at them separately.

If you think about left/right or up/down directions, or if you use phase-plane software, you will notice that the arrowheads on the slope marks on the far left of our window have to point in the opposite direction from those on the right.

Furthermore, if you seek equilibria by setting \( x' = 0 \) and \( y' = 0 \), you will see that equilibria cover the entire half-line \( x = -1 \), separating the trajectories heading to the upper right from those heading to the lower left. Notice that this does not contradict the statements above re existence and uniqueness, but rather emphasizes the wisdom of going as far as you can with analyzing a phase portrait. In particular, it shows that attention to direction arrows is essential to proper mastery of a system.
40. \[ x' = \frac{x}{y} = f(x, y) \]
\[ y' = x - \frac{y}{x} = g(x, y) \]

(a) The domain of the system of DEs is the entire \( xy \)-plane except the \( x \)- and \( y \)-axes. Because \( f \) is continuous except when \( y = 0 \), and \( g \) is continuous except when \( x = 0 \), there exists a solution passing through each point \((x_0, y_0)\) not on either axis.

The partial derivatives
\[ \frac{\partial f}{\partial x} = \frac{1}{y} \quad \frac{\partial f}{\partial y} = -\frac{x}{y^2} \]
\[ \frac{\partial g}{\partial x} = 1 + \frac{y}{x^2} \quad \frac{\partial g}{\partial y} = -\frac{1}{x} \]
are continuous as long as \( x \neq 0 \), and \( y \neq 0 \), so there exists a unique solution passing through any point not on either coordinate axis.

(b) The figure for the phase plane and sample trajectories shows that there seems to be a unique solution passing through each point that is not on one of the coordinate axes. At the origin the DE is not defined, but trajectories in the second and fourth quadrants seem to treat the origin as a stable node, while those in the first and third quadrants treat it like a saddle.

### Hamiltonian for the Harmonic Oscillator

41. (a) Introducing \( q = x \) and \( p = mx \), the kinetic energy of the undamped harmonic oscillator \( mx + kx = 0 \) can be written as
\[ KE = \frac{1}{2} mx^2 = \frac{1}{2} m \left(\frac{p}{m}\right)^2 = \frac{p^2}{2m} \]

(b) The total energy of the undamped harmonic oscillator is the sum of the kinetic and potential energies. Because
\[ PE = \frac{1}{2} kx^2 = \frac{1}{2} kq^2 \]
the total energy in generalized coordinates \((q = x \text{ and } p = mx)\) becomes
\[ H(p, q) = \frac{p^2}{2m} + \frac{kq^2}{2} \]

Here \( \frac{p^2}{2m} \) is the kinetic energy and \( \frac{kq^2}{2} \) the potential energy.

(c) The Hamiltonian system is
\[ \dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m} \]
\[ \dot{p} = -\frac{\partial H}{\partial q} = -kq. \]
Computer Lab: Phase Plane Analysis

Computer phase portraits, as in Problems 42-47, must clearly indicate the directions of the trajectories, with arrows visible either on the vector field or on the trajectories. For the tiny figures in this manual we have chosen the latter as being the most easily readable.

42. \[ x' = x(x - y) \]
\[ y' = y(1 - y) \]

(a),(b) See the figure for the vector field with sample trajectories (and the v-nullcline).
(c),(d) The equilibria are at (0,0), (0,1), (1,1). The figure shows that (0,1) is stable and attracting; (1,1) is a saddle (unstable); (0,0) is unstable in an interesting way (a saddle on the left and a node on the right!).
(e),(f) The long-term behavior of this system depends on the initial conditions. Most trajectories starting in the upper half plane head toward (0,1), while some are deflected by the saddle on the right and move toward \( y = 1 \) then toward infinity. Trajectories starting in the lower half plane move downward toward negative infinity.

The phase portrait does not detect any periodic solutions, which would appear as closed loop trajectories.

43. \[ x' = x - x^2 \]
\[ y' = -y \]

(a),(b) See the figure for the vector field of this system, with sample trajectories.
(c),(d) The equilibria are at (0,0) and (1,0). The figure shows that (0,0) is unstable (a saddle) and (1,0) is stable. The latter however is unlike the standard linear node:
   • on the left the trajectories tend towards \(-\infty\) along the horizontal axis;
   • for \( 0 < x < 1 \) trajectories come in rather vertically from \( \pm \infty \) and approach (1,0);
   • for \( x > 1 \) trajectories flow from the right more directly toward (1,0).
(e) The long-term behavior of this system depends on the initial conditions. For \( x > 0 \), trajectories move toward the stable equilibrium at (1,0). For \( x < 0 \), trajectories approach the \( x\)-axis and go off to \(-\infty\).
(f) The phase portrait does not detect any periodic solutions, which would appear as closed loop trajectories.
44. \[ x' = 1 - |x| \]
\[ y' = x - y \]

(a),(b) See the figure for the vector field of this system, with sample trajectories.

(c),(d) The equilibria are at (1,1) and (-1,-1). The figure shows that (1,1) is stable; (-1,-1) is a saddle.

(e) The long-term behavior of this system depends on the initial conditions. All trajectories to the right of \( x = -1 \) move toward the stable equilibrium at (1,1). All trajectories to the left of \( x = -1 \) head toward \( -\infty \) in both \( x \) and \( y \) directions.

(f) The phase portrait does not detect any periodic solutions, which would appear as closed loop trajectories.

45. \[ x' = x(2 - x - y) \]
\[ y' = -y \]

(a),(b) See the figure for the vector field, plus the v-nullcline, with sample trajectories.

(c),(d) The equilibria are at (0,0) and (2,0). The figure shows that (0,0) is unstable (a saddle) and (2,0) is stable.

(e) The long-term behavior of this system depends on the initial conditions. For \( x > 0 \), trajectories move toward the stable equilibrium at (2,0). For \( x < 0 \), trajectories approach the \( x \)-axis and go off to \( -\infty \).

(f) The phase portrait does not detect any periodic solutions, which would appear as closed loop trajectories.

46. \[ x' = x + y - x^3 \]
\[ y' = -x \]

(a),(b) See the figure for the vector field of this system, with sample trajectories.

(c),(d) The only equilibrium is at (0,0). The figure shows that (0,0) is an unstable spiral point.

(e),(f) The phase portrait shows a periodic solution, which is a limit cycle because it attracts trajectories from all initial points except the origin.
47. \[ x' = \sin(xy) \]
\[ y' = \cos(x + y) \]

(a), (b) See the first figure for a small phase portrait \((-4 \leq x \leq 4, -4 \leq y \leq 4\) of this system, with sample trajectories. The cover of this manual shows a larger region \((-10 \leq x \leq 10, -10 \leq y \leq 10\).

(c), (d) The v-nullclines are the hyperbolae
\[ xy = \pm n\pi \] for integer \(n\).

The h-nullclines are the lines
\[ x + y = \pm (2n + 1)\pi \] or integer \(n\)

These nullclines intersect to create many equilibrium points (see second figure). Those visible in the phase portrait are as follows:

A \((-1.5708, 0.0000)\) Unstable spiral
B \((0.0000, 1.5708)\) Unstable saddle
C \((1.5708, 0.0000)\) Stable spiral
D \((0.0000, -1.5708)\) Unstable saddle
E \((-2.7241, 1.1533)\) Unstable saddle
F \((-1.1533, 2.7241)\) Stable spiral
G \((2.7241, -1.1533)\) Unstable saddle
H \((1.1533, -2.7241)\) Unstable spiral
I \((-3.9086, -0.8038)\) Stable spiral
J \((0.83038, 3.9086)\) Unstable saddle
K \((3.9086, 0.8038)\) Unstable spiral
L \((-0.83038, -3.9086)\) Unstable saddle
M \((-3.4122, 1.8414)\) Unstable spiral
N \((-1.8414, 3.4122)\) Unstable saddle
O \((3.4122, -1.8414)\) Stable spiral
P \((1.8414, -3.4122)\) Unstable saddle
Q \((-3.9543, 2.3835)\) Unstable saddle
R \((-2.3835, 3.9543)\) Stable spiral
S \((3.9543, -2.3835)\) Unstable saddle
T \((2.3835, -3.9543)\) Unstable spiral

Continued on next page.
(e)(f) The long-term behavior of this system is very complex and depends on the initial conditions. See figures and text for (c),(d). In the window shown in parts (a)-(d), the phase portrait does not detect any periodic solutions or closed loop trajectories. 

But in a larger window we can find closed loops! The next figure increases the window bounds sufficiently to show some sausage-shaped limit cycles in the first and third quadrants. In the final figure (a closer look at the first quadrant, showing only two saddles with their separatrices), we give a clearer view of some of these limit cycles.

Zooming out to catch some limit cycles for

\[ x' = \sin(xy), \quad y' = \cos(x + y). \]

Saddles, at approximately (2.0, 9.5) and (9.5, 2.0), with separatrices showing stable limit cycles (each enclosing an unstable spiral equilibrium not drawn).

- **Computer Lab: Graphing in Two Dimensions**
  - Student Lab Projects with IDE
- **Computer Lab: The Glider**
  - Student Lab Projects with IDE
- **Computer Lab: Nonlinear Oscillators**
  - Student Lab Projects with IDE
- **Suggested Journal Entry**
  - Student Project
7.2 Linearization

Review of Classifications

At a given nonlinear equilibrium of a system \( x' = f(x, y), \ y' = g(x, y) \), we can use the Jacobian matrix

\[
J = \begin{bmatrix}
  f_x(x, y) & f_y(x, y) \\
  g_x(x, y) & g_y(x, y)
\end{bmatrix}
\]

to quickly and algebraically calculate the stability, by finding either

- the eigenvalues of \( J \) (Reference: Table 7.2.1 in the text)
- or

- the location of the trace and determinant of \( J \) in the linear classification diagram. (Reference: Figure 7.2.7 in the text)

Occasionally for a nonlinear equilibrium further analysis is still necessary.

- If the linearization is a center, the nonlinear equilibrium could be a center or a spiral of either stability.
- Sometimes an equilibrium is a combination, due to a linearization that is a borderline case (i.e., degenerate). E.g., you could find a nonlinear equilibrium to be a saddle on one side and a node on the other.
- Sometimes there is a whole line or curve of equilibria.

For each of Problems 1-19 we show various ways of reaching the conclusions – a phase portrait of the nonlinear system, the linearizations about each equilibrium, each with its Jacobian and stability analysis, and sometimes small zooms of these linearizations. **Focus on the elements that work best for you, in whatever order.**

### Original Equilibrium

1. \( x' = -2x + 3y + xy \)
   \( y' = -x + y - 2xy^2 \)

At \((0,0)\) \( x' = y' = 0 \), so the origin is an equilibrium point. The phase portrait, linearization, and Jacobian calculations are shown.

At \((0,0)\)

\[
J = \begin{bmatrix}
  -2 & 3 \\
  -1 & 1
\end{bmatrix}
\]

\[
\text{tr}J = -1, \ \det J = 1
\]

\[
\lambda = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i
\]

\[
\approx -0.5 \pm 0.87i
\]

We conclude that \((0,0)\) is a stable spiral (complex eigenvalues with negative real part).
2. \[ x' = -y - x^3 \quad \quad y' = x - y^3 \]

\[ J(x, y) = \begin{bmatrix} -3x^2 & -1 \\ 1 & -3y^2 \end{bmatrix} \]

At \((0,0)\) \(x' = y' = 0\), so the origin is an equilibrium point. The phase portrait, linearization, and Jacobian calculations are shown.

The linearization finds the equilibrium to be a center, but that is a borderline case and the nonlinear equilibrium could be a center or a spiral of either stability. The phase portrait shows that trajectories spiral toward the origin, but so slowly they seem to leave a hole in the middle. We must check that trajectories are not just attracted to a small limit cycle instead of the origin. By zooming in, or just letting the trajectory run longer to see if the trajectory will fill in the center space, we can conclude that the origin is an asymptotically stable spiral equilibrium for the nonlinear system, but a very weak attractor. We conclude that \((0,0)\) is a stable spiral (complex eigenvalues with negative real part).

Zoom on the origin for \(x' = -y - x^3, \ y' = x - y^3\)
3. \[ x' = x + y + 2xy \quad \quad J(x, y) = \begin{bmatrix} 1 + 2y & 1 + 2x \\ -2 & 1 + 3y^2 \end{bmatrix} \]

At \((0,0)\) \(x' = y' = 0\), so the origin is an equilibrium point. The phase portrait, linearization, and Jacobian calculations are shown.

We conclude that \((0,0)\) is an unstable spiral (complex eigenvalues with positive real part). We also know that \((-1, -1)\) is a saddle (real eigenvalues of opposite sign).

4. \[ x' = y \quad \quad J(x, y) = \begin{bmatrix} 0 & 1 \\ -\cos x & -1 \end{bmatrix} \]

At \((0,0)\) \(x' = y' = 0\), so the origin is an equilibrium point. But there are other equilibria at \((\pm n\pi, 0)\) for any integer \(n\). We will investigate those that appear in the phase portrait shown.

We conclude that \((0,0)\) is a center (purely imaginary eigenvalues, symmetry of nonlinear direction field about both axes). We also conclude that \((\pm \pi, 0)\) are both saddles (real eigenvalues of opposite sign).
5. \[ x' = x + y^2 \]
\[ y' = x^2 + y^2 \]
\[ J(x, y) = \begin{bmatrix} 1 & 2y \\ 2x & 2y \end{bmatrix} \]
At \((0,0)\) \(x' = y' = 0\), so the origin is an equilibrium point. But there are other equilibria at \((\pm n\pi, 0)\) for any integer \(n\). We will investigate those that appear in the phase portrait shown.

The linearization is a degenerate case, with an entire line of unstable equilibria along the \(y\)-axis. The trace and determinant of the Jacobian place this example on the boundary between unstable node and saddle. (See Figure 7.2.7 in the text.) The zero eigenvalue creates no direction.

We see that the nonlinear equilibrium is an unstable combination: a saddle for \(y < 0\), and an unstable node for \(y > 0\).

6. \[ x' = \sin y^2 \]
\[ y' = -\sin x + y \]
\[ J(x, y) = \begin{bmatrix} 0 & \cos y \\ -\cos x & 1 \end{bmatrix} \]
At \((0,0)\) \(x' = y' = 0\), so the origin is an equilibrium point. There are other equilibria at \((\pm n\pi, 0)\) for any integer \(n\). We will investigate those that appear in the phase portrait shown.

We conclude that \((0,0)\) is an unstable spiral (complex eigenvalues with positive real part). The equilibria at \((\pm \pi, 0)\) are both saddles (real eigenvalues of opposite sign), while those at \((0, \pm \pi)\) are both unstable spirals (complex eigenvalues with positive real part). We leave it to the reader to make similar calculations and conclusions for other equilibria.
Unusual Equilibria

7. \[ x' = 1 - xy \]
\[ y' = x - y^3 \]

\[ J(x, y) = \begin{bmatrix} -y & -x \\ 1 & -3y^2 \end{bmatrix} \]

We find equilibria at (1,1) and (-1,-1). The phase portrait, linearizations, and Jacobian calculations are shown.

At (1,1) \[ J = \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix} \]

\[ \text{tr} J = -4, \ \text{det} J = 4 \]

double eigenvalue \( \lambda = -2 \)

At (-1,-1) \[ J = \begin{bmatrix} 1 & 1 \\ 1 & -3 \end{bmatrix} \]

\[ \text{tr} J = -2, \ \text{det} J = -4 \]

\[ \lambda = -1 \pm \sqrt{5} \]

We conclude that (1,1) is stable (a borderline case in the trace-determinant plane with double and negative eigenvalue), and that (-1, -1) is a saddle (real eigenvalues of opposite sign). The equilibrium information is sufficient to determine all directions in the phase portrait.

8. \[ x' = x - 3y + 2xy \]
\[ y' = 4x - 6y - xy \]

\[ J(x, y) = \begin{bmatrix} 1 + 2y & -3 + 2x \\ 4 - y & -6 - x \end{bmatrix} \]

We find equilibria at (0,0) and (2/3,2/5). The phase portrait, linearizations, and Jacobian calculations are shown.

At (0,0) \[ J = \begin{bmatrix} 1 & -3 \\ 4 & -6 \end{bmatrix} \]

\[ \text{tr} J = -5, \ \text{det} J = 6 \]

\( \lambda_1 = -2, \ \lambda_2 = -3 \)

At (2/3,2/5) \[ J = \begin{bmatrix} 9/5 & -5/3 \\ 18/5 & -20/3 \end{bmatrix} \]

\[ \text{tr} J = -73/15, \ \text{det} J = 90/15 \]

\( \lambda = (-1 \pm \sqrt{5})/2 \)

We conclude that (0,0) is an asymptotically stable node (real eigenvalues, both negative), and that (2/3, 2/5) is a saddle (real eigenvalues of opposite sign). The equilibrium information is sufficient to determine all directions in the phase portrait.
9. \( x' = 4x - x^3 - xy^2 \)
   \( y' = 4y - x^2y - y^3 \)

\[ J(x, y) = \begin{bmatrix}
4 - 3x^2 - y^2 & -2xy \\
-2xy & 4 - x^2 - 3y^2
\end{bmatrix} \]

We find equilibria at \((0,0)\) and \((2/3,2/5)\). The phase portrait, linearizations, and Jacobian calculations are shown.

We conclude that the origin is an unstable node (real eigenvalues, both positive), and could easily be fooled into thinking that was the end of the story. But, a sharper eye either to the algebra or to computer trajectories shows there is more – a whole circle of equilibria where \( x^2 + y^2 = 4 \).

Because each point on the circle has different coordinates, it is difficult to examine the stability of these equilibrium points using the Jacobian. However if from the nonlinear system we write

\[ \frac{dy}{dx} = \frac{y'}{x'} = \frac{y}{x'} \]

we can see that the trajectories in the phase plane are simply straight lines \( y = cx \). Analysis of the signs of \( x' \) and \( y' \) shows by the quadratic factor that movement is always toward the circle \( x^2 + y^2 = 2 \). Hence, any trajectory beginning at an initial condition outside the circle will move towards the circle. The points on the circle are stable equilibria. In the final figures we show sample linearizations for two such equilibrium points.

At \((0.953, 1.759)\)

\[ J \approx \begin{bmatrix}
-1.8 & -3.35 \\
-3.35 & -6.2
\end{bmatrix} \]

\( \text{tr} J = -8, \det J = 0 \)

\( \lambda_1 = 0, \lambda_2 = -8 \)

At \((1.275, -1.541)\)

\[ J \approx \begin{bmatrix}
-3.25 & 3.93 \\
3.93 & -4.75
\end{bmatrix} \]

\( \text{tr} J = -8, \det J = 0 \)

\( \lambda_1 = 0, \lambda_2 = -8 \)
### Linearization Completion

10. \[ x' = y \]
    \[ y' = -y + x - x^3 \]
    \[ J(x,y) = \begin{bmatrix} 0 & 1 \\ 1 - 3x^2 & -1 \end{bmatrix} \]

Setting \( x' = y' = 0 \) we find equilibria at (0,0) and (-1,0), and analyze them as follows.

At (0,0)

\[ J = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \]
\[ \text{tr} J = -1, \quad \text{det} J = -1 \]
\[ \lambda = (-1 \pm \sqrt{5})/2 \]

Saddle

At (±1,0)

\[ J = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix} \]
\[ \text{tr} J = -1, \quad \text{det} J = -1 \]
\[ \lambda = (-1 \pm \sqrt{7}i)/2 \]

Stable spirals

---

### Uncertainty

11. \[ x^3 + x^2 + x + x^3 = 0 \] can be written as a system

\[ x' = y \]
\[ y' = -y - x - x^3 \]

Setting \( x' = y' = 0 \) we find a single equilibria at (0,0).

At (0,0)

\[ J = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \]
\[ \text{tr} J = -1, \quad \text{det} J = 1 \]
\[ \lambda = (-1 \pm \sqrt{3}i)/2 \]

Stable spiral

12. \[ x^3 + x^2 + x + x^3 = 0 \] can be written as a system

\[ x' = y \]
\[ y' = -y - x + x^3 \]

Setting \( x' = y' = 0 \) we find a single equilibria at (0,0), (1,0) and (-1,0) which we analyze as follows.

At (0,0)

\[ J = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \]
\[ \text{tr} J = -1, \quad \text{det} J = 1 \]
\[ \lambda = (-1 \pm \sqrt{3}i)/2 \]

Stable spiral

At (±1,0)

\[ J = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} \]
\[ \text{tr} J = -1, \quad \text{det} J = -2 \]
\[ \lambda_1 = -2, \quad \lambda_2 = 1 \]

Saddles
Liénard Equation

13. \( \ddot{x} + p(x) \dot{x} + q(x) = 0 \)

Letting \( \dot{x} = y \), we obtain the nonlinear system

\[
\begin{align*}
\dot{x} &= y = f(x, y) \\
\dot{y} &= -p(x)y - q(x) = g(x, y)
\end{align*}
\]

for which the Jacobian is

\[
J(x, y) = \begin{bmatrix} 0 & 1 \\
-p'(x)y - q'(x) & -p(x) \end{bmatrix}
\]

We are given \( q(0) = 0 \) and \( p(0) > 0 \), so \( \dot{x}(0) = 0 \) and \( \dot{y}(0,0) = 0 \). The origin is an isolated equilibrium point with

\[
J(0,0) = \begin{bmatrix} 0 & 1 \\
-q'(0) & -p(0) \end{bmatrix}
\]

which is nonsingular because \( q'(0) > 0 \) is also given. The characteristic equation

\[
\lambda^2 + p(0)\lambda + q'(0) = 0
\]

has roots

\[
\lambda = \frac{-p(0) \pm \sqrt{p(0)^2 - 4q'(0)}}{2}
\]

that are either negative or have negative real parts. Hence, \( (0,0) \) is a stable equilibrium point.

Conservative Equation

14. \( x'' + x - x^2 + 2x^3 = 0 \) can be written as

\[
\begin{align*}
x' &= y \\
y' &= -x + x^2 + 2x^3
\end{align*}
\]

with \( J(x, y) = \begin{bmatrix} 0 & 1 \\
-1 + 2x + 6x^2 & 0 \end{bmatrix} \)

Setting \( x' = y' = 0 \) we find equilibria at \( (0,0), (1/2,0) \) and \( (-1,0) \), which we analyze as follows.

At \( (0,0) \) \( \text{tr}J = 0, \text{det}J=1; \lambda = \pm i \)

Center (not spiral, by symmetry of direction field)

At \( (1/2,0) \) \( \text{tr}J = 0, \text{det}J=-3; \lambda = \pm \sqrt{3} \)

Saddle

At \( (1/2,0) \) \( \text{tr}J = 0, \text{det}J=-3/2; \lambda = \pm 3/2 \)

Saddle
CHAPTER 7  Nonlinear Systems of Differential Transformations

Predator-Prey Equations

15. \( x' = (a - by)x = f(x, y) \)

\( y' = (cx - d)y = g(x, y) \)

(a) The Jacobian of this system is

\[
J(x, y) = \begin{bmatrix}
    f_x & f_y \\
    g_x & g_y
\end{bmatrix} = \begin{bmatrix}
    a - by & -bx \\
    cy & cx - d
\end{bmatrix}
\]

so at the equilibrium point \( \left( \frac{d}{c}, \frac{a}{b} \right) \)

\[
J = \begin{bmatrix}
    0 & \frac{-bd}{c} \\
    \frac{ac}{b} & 0
\end{bmatrix}, \text{ with eigenvalues } \pm i\sqrt{ad}.
\]

Hence, the equilibrium point \( \left( \frac{d}{c}, \frac{a}{b} \right) \) could be either a center or a spiral of unknown stability.

The phase plane portrait for this system when \( a = b = c = d = 1 \) shows the equilibrium point \( (1,1) \) as a center. (See the answer for Problem 9 in section 2.6.)

Van der Pol's Equation

16. \( \ddot{x} - \varepsilon(1 - x^2)\dot{x} + x = 0 \)

Letting \( y = \dot{x} \), write van der Pol's equation as

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= -x + \varepsilon(1 - x^2)y
\end{align*}
\]

which has a single equilibrium point at the origin. Linearizing this system by dropping the higher-order terms in \( x \) and \( y \) yields

\[
\begin{bmatrix}
    \dot{x} \\
    \dot{y}
\end{bmatrix} = \begin{bmatrix}
    0 & 1 \\
    -1 & \varepsilon
\end{bmatrix} \begin{bmatrix}
    x \\
    y
\end{bmatrix},
\]

which has eigenvalues

\[\lambda_1, \lambda_2 = \frac{\varepsilon \pm \sqrt{\varepsilon^2 - 4}}{2}\]

Hence, for any \( \varepsilon > 0 \) there are either real eigenvalues with one positive eigenvalue, or complex eigenvalues with positive real part. hence, the origin \( (0,0) \) is always an unstable equilibrium point of the nonlinear system. See figure for trajectories of van der Pol's equation for \( \varepsilon = 1 \), showing a stable limit cycle surrounding the origin.
## Damped Mass-Spring Systems

17. \(\ddot{x} + \dot{x}^3 + x = 0\)

Writing the equations as a system yields

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= -x - y^3
\end{align*}
\]

that has the single equilibrium point \((0,0)\). The linearized equation about this point is

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= -x
\end{align*}
\]

whose Jacobian at \((0,0)\) is

\[
J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}
\]

The eigenvalues of this Jacobian are \(\pm i\), and so the origin of the nonlinear system could be either a center or a spiral point of unknown stability. From the figure for trajectories of this system, the origin appears to be a stable spiral.

18. \(\ddot{x} + \dot{x} - \dot{x}^3 + x = 0\)

Writing the equation as a system yields

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= -x - y^3 + y^3
\end{align*}
\]

Setting \(\dot{x} = \dot{y} = 0\) yields the equilibrium point \((0,0)\). The linearized equations about this point are

\[
\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},
\]

which has eigenvalues

\[
\lambda_1, \lambda_2 = -\frac{1}{2} \pm i \frac{\sqrt{3}}{2}
\]

Hence, the origin is an asymptotically stable spiral point of the nonlinear system. Intuition suggests the zero solution \(x = \dot{x} = 0\) is stable because in a neighborhood of zero the positive damping term \(\dot{x}\) is larger than the negative damping term \(-x^3\).

The phase portrait for this nonlinear system shows a periodic solution (limit cycle) that is unstable.
19. \( \ddot{x} + \dot{x} + x^3 + x = 0 \)

Writing the equation as a system yields

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= -x - y - y^3
\end{align*}
\]

Setting \( \dot{x} = \dot{y} = 0 \) yields the equilibrium point \((0,0)\). The linearized equations about this point are

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= -x - y
\end{align*}
\]

whose Jacobian at \((0,0)\) is

\[
J = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}
\]

The eigenvalues of this matrix are the complex numbers

\[
\lambda = -\frac{1}{2} \pm i \frac{\sqrt{3}}{2},
\]

so the origin of the nonlinear system is a stable spiral. See figure for sample trajectories of this system spiraling towards the origin.

20. \( \ddot{x} - \dot{x} + x = 0 \)

The equation can be written as the first-order system

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= -x + y
\end{align*}
\]

or

\[
\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.
\]

Note that the system is already linear. The system has negative damping so we suspect the origin is stable. The eigenvalues of the system around \((0,0)\) are

\[
\lambda_1, \lambda_2 = -\frac{1}{2} \pm i \frac{\sqrt{3}}{2},
\]

which confirms that the equilibrium point is an unstable spiral point.
Liapunov Functions

21. \[ \begin{align*}
x' &= y - 2x^3 \\
y' &= -2x - 3y^5
\end{align*} \]
with \( L(x, y) = 2x^2 + y^2 \)

Solving \( x' = y' = 0 \), we see that (0,0) is an equilibrium, and analysis of the linearization tells us it could be either a center or a spiral, of either stability. The phase portrait does not readily answer the question, because the trajectories go ever more slowly as they approach the origin; a legitimate question is whether there is a limit cycle surrounding the origin. Liapunov's direct method gives a quick answer.

The given function \( L(x,y) \) is clearly positive definite. Furthermore, calculation shows that
\[
\frac{dL}{dt} = L_x x' + L_y y' = 4xx' + 2yy' = 4x(y - 2x^3) + 2y(-2x - 3y^5) = -(8x^4 + 6y^6)
\]
is negative definite. Hence Liapunov's result tells us that the origin is asymptotically stable and there is no limit cycle.

22. \[ \begin{align*}
x' &= 2y - x^3 \\
y' &= -x^3 - y^5
\end{align*} \]
with \( L(x, y) = x^4 + 4y^2 \)

Solving \( x' = y' = 0 \), we see that (0,0) is an equilibrium, and analysis of the linearization tells us it could be either a center or a spiral. The phase portrait does not definitively reveal which, again because the trajectories go ever more slowly as they approach the origin.

The given function \( L(x,y) \) is positive definite, and calculation shows that
\[
\frac{dL}{dt} = L_x x' + L_y y' = 4x^3(2y - x^3) + 8y(-x^3 - y^5) = 8x^3y - 4x^6 - 8x^3y - 8y^6 = (4x^6 + 8y^6)
\]
is negative definite. Hence by Liapunov's theorem we know the origin is asymptotically stable.
A Bifurcation Point

23. \( \dot{x} = -x(y^2 + 1) \)
   \( \dot{y} = y^2 + k \)

(a) Setting \( \dot{x} = \dot{y} = 0 \), yields
   \(-x(y^2 + 1) = 0 \) and \( y^2 + k = 0 \).
   Looking for real roots, the first equation yields \( x = 0 \)
   and when \( k > 0 \), the second equation yields \( y = \pm \sqrt{-k} \).
   Hence we have two equilibrium points
   \((0, \sqrt{-k})\) and \((0, -\sqrt{-k})\).

(b) When \( k = 0 \), we have the root \( x = y = 0 \) and hence the
   single equilibrium point \((0,0)\).

(c) When \( k > 0 \), the second equation \( y^2 + k = 0 \) clearly
   has no real root, so there are no equilibria.

(d) When \( k = 0 \), the linearized system can be found by
   simply dropping the higher order terms, yielding
   \( \dot{x} = -x \) and \( \dot{y} = 0 \). As the final figure shows, this
   linearization at the bifurcation value has a whole line
   of stable equilibrium points. I.e., the linearization is a
   borderline case.
Computer Lab: Trajectories

24. \( x'' + x \sin x = 0 \)

Letting \( y = \dot{x} \), the equation can be written as the first-order system

\[
\begin{align*}
x' &= y = f(x, y) \\
y' &= -x \sin x = g(x, y)
\end{align*}
\]

The phase plane trajectories of this system can be studied by looking at the direction field of

\[
\frac{dy}{dx} = \frac{y'}{x'} = \frac{-x \sin x}{y}
\]

in the \( xy \) plane. See figure.

The nonlinear system has equilibria at the points \((\pm n\pi, 0)\). We examine the stability of the nonlinear system at the three equilibria \((\pm \pi, 0)\) and \((0, 0)\). The Jacobian of the system is

\[
J = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\sin x - x \cos x & 0 \end{bmatrix}
\]

and so the Jacobians at the three equilibrium points are

\[
J(0, 0) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad J(-\pi, 0) = \begin{bmatrix} 0 & 1 \\ -\pi & 0 \end{bmatrix} \quad J(\pi, 0) = \begin{bmatrix} 0 & 1 \\ \pi & 0 \end{bmatrix}
\]

At \((0, 0)\) the Jacobian is singular indicating that the linear system \((x' = y, y' = 0)\) does not have an isolated equilibria at the origin. Hence, the nonlinear system cannot be linearized about the origin and shows nothing about the stability of the nonlinear system at \((0, 0)\). See figure for phase drawing that indicates \((0, 0)\) is an unstable equilibrium point.

At \((\pm \pi, 0)\) the Jacobian of the linearized system has eigenvalues \(\lambda = \pm i \sqrt{\pi}\); hence, the nonlinear system has either a center or stable or unstable spiral point at \((\pm \pi, 0)\). The figure shows it to be a center.

At \((\pi, 0)\) the Jacobian of the linearized system has eigenvalues \(\lambda = \pm \sqrt{\pi}\), and so the nonlinear system has an unstable saddle at \((\pi, 0)\), as shown in the figure.

In fact, as we see from the figure, all along the \(x\)-axis the equilibria at multiples of \(\pi\) alternate between saddles and centers.
25. \( x'' + x - 0.1(x^2 + 2x^3) = 0 \)

Letting \( y = x' \), the equation can be written as the first-order system

\[
\begin{align*}
x' &= y \\
y' &= -x + 0.1(x^2 + 2x^3)
\end{align*}
\]

and whose trajectories in the phase plane can be studied by looking at the direction field of

\[
\frac{dy}{dx} = \frac{y'}{x'} = -\frac{x - 0.1(x^2 + 2x^3)}{y}
\]

in the \( xy \) plane. Note that the trajectories cross the \( x \)-axis in a vertical manner due to the \( y \) in the denominator of \( \frac{dy}{dx} \).

If the system is linearized around \((0,0)\) it yields

\[
\begin{align*}
x' &= y \\
y' &= -x
\end{align*}
\]

whose trajectories are the circles shown in the figure. There are two more equilibria at \( \left( -\frac{5}{2}, 0 \right) \) and \((2,0)\), whose linearizations are saddle points.

26. \( x'' - (1 - x^2)x' + x = 0 \)

Letting \( y = x' \), the equation can be written as the first-order system

\[
\begin{align*}
x' &= y \\
y' &= (1 - x^2)y - x
\end{align*}
\]

and whose trajectories in the phase plane can be studied by looking at the direction field of

\[
\frac{dy}{dx} = \frac{y'}{x'} = -\frac{(1 - x^2)y - x}{y}
\]

in the \( xy \) plane. Note that the trajectories cross the \( x \)-axis in a vertical manner due to the \( y \) in the denominator of \( \frac{dy}{dx} \).

If the system is linearized around \((0,0)\) it yields

\[
\begin{align*}
x' &= y \\
y' &= -x + y
\end{align*}
\]

whose trajectories are shown in the figure. Note the resemblance between the trajectories of the nonlinear system and the linear system close to the origin.

Also note that the nonlinear portrait shows a limit cycle.
27. \( x'' + x - 0.25x^3 = 0 \)

Letting \( y = x' \), the equation can be written as the first-order system

\[
\begin{align*}
x' &= y \\
y' &= -x + 0.25x^2
\end{align*}
\]

having equilibrium points (0,0) and (4,0) whose trajectories in the phase plane can be studied by looking at the direction field of

\[
\frac{dy}{dx} = \frac{y'}{x'} = -\frac{x + 0.25x^2}{y}
\]

in the \( xy \) plane. Note that the trajectories cross the \( x \)-axis in a vertical manner due to the \( y \) in the denominator of \( \frac{dy}{dx} \).

The linearized equation at equilibrium (0,0) are

\[
\begin{align*}
x' &= y \\
y' &= -x
\end{align*}
\]

whose trajectories are the circles drawn in Problem 25. The linearization at (4,0) can also be determined and classified.

- **Computer Lab: Competition**

28. Student Lab Projects with IDE

- **Suggested Journal Entry I**

29. Student Project

- **Suggested Journal Entry II**

30. Student Project
7.3 Numerical Solutions

**Spreadsheet Calculation**

1. \( x' = y \quad x(0) = 1 \)
   \( y' = -x + x^3 - y \quad y(0) = 1 \)

The following instructions show one way to carry out Euler's method on most spreadsheets. After entering the initial conditions and the formulas (cell entries that begin with "="), simply pull down the cells with the mouse for the spreadsheet to "fill" the proper numbers, as shown in the table that follows.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>t</td>
<td>x</td>
<td>y</td>
<td>xdot</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>=C2</td>
</tr>
<tr>
<td>3</td>
<td>=A2+F2</td>
<td>=B2+F2xD2</td>
<td>=C2+F2xE2</td>
<td>=C3</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>=C2</td>
</tr>
<tr>
<td>5</td>
<td>=A2+F2</td>
<td>=B2+F2xD2</td>
<td>=C2+F2xE2</td>
<td>=C3</td>
</tr>
</tbody>
</table>

Spreadsheet instructions with Euler's method for \( x' = y, \ y' = -x + x^3 - y \)

We can calculate the numerical approximation at \( t = 1 \) for Euler’s method using \( h = 0.05 \) by just changing cell F2, then dragging down all of row 3 until \( t = 1 \). We tabulate the results as follows.

- \( h = 0.1 \)
  - \( x(1) = 1.9596 \)
  - \( y(1) = 1.6559 \)
- \( h = 0.05 \)
  - \( x(1) = 2.0126 \)
  - \( y(1) = 1.8701 \)

The phase plane figure compares the results with the two step sizes for \( 0 \leq t \leq 1 \), starting at the initial condition \((1,1)\).

Note that the smaller step size allows Euler's method to better follow solutions around curves, and it gives higher values for both \( x(1) \) and \( y(1) \).

We can expect the exact solution for \( t = 1 \) to be close to the approximate solution when \( h = 0.05 \), with slightly higher values for \( x(1) \) and \( y(1) \).
2. \[ x' = y \quad x(0) = 1 \]
\[ y' = -x - x^3 - y \quad y(0) = 1 \]

We solve this initial value problem exactly as in Problem 1, except we now enter the different equation for \( y' \) as the command "=–B2–B2^3–C2" in cell E2, and drag down the cells again.

Note also that initial conditions and/or step size can be changed by entering new numbers in their respective cells and pulling down those cells. The results for \( h = 0.1 \) are shown.

<table>
<thead>
<tr>
<th>t</th>
<th>x</th>
<th>y</th>
<th>xdot</th>
<th>ydot</th>
<th>h</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-3</td>
<td>0.1</td>
</tr>
<tr>
<td>0.1</td>
<td>1.1</td>
<td>0.7</td>
<td>0.7</td>
<td>-3.131</td>
<td>0.1</td>
</tr>
<tr>
<td>0.2</td>
<td>1.17</td>
<td>0.3869</td>
<td>0.4</td>
<td>-3.1585</td>
<td>0.1</td>
</tr>
<tr>
<td>0.3</td>
<td>1.2087</td>
<td>0.07105</td>
<td>0.1</td>
<td>-3.0456</td>
<td>0.1</td>
</tr>
<tr>
<td>0.4</td>
<td>1.2158</td>
<td>-0.2335</td>
<td>0</td>
<td>-2.7794</td>
<td>0.1</td>
</tr>
<tr>
<td>0.5</td>
<td>1.1924</td>
<td>-0.5114</td>
<td>-1</td>
<td>-2.3766</td>
<td>0.1</td>
</tr>
<tr>
<td>0.6</td>
<td>1.1413</td>
<td>-0.7491</td>
<td>-1</td>
<td>-1.8788</td>
<td>0.1</td>
</tr>
<tr>
<td>0.7</td>
<td>1.0664</td>
<td>-0.937</td>
<td>-1</td>
<td>-1.3421</td>
<td>0.1</td>
</tr>
<tr>
<td>0.8</td>
<td>0.9727</td>
<td>-1.0712</td>
<td>-1</td>
<td>-0.8218</td>
<td>0.1</td>
</tr>
<tr>
<td>0.9</td>
<td>0.8656</td>
<td>-1.1534</td>
<td>-1</td>
<td>-0.3607</td>
<td>0.1</td>
</tr>
<tr>
<td>1</td>
<td>0.7502</td>
<td>-1.1894</td>
<td>-1</td>
<td>0.01694</td>
<td>0.1</td>
</tr>
</tbody>
</table>

Spreadsheet calculations by Euler’s method for \( h = 0.1 \)

We can calculate the numerical approximation at \( t=1 \) for Euler’s method using \( h=0.05 \) by just changing cell F2, then dragging down all of row 3 until \( t = 1 \). The results can be compared as follows:

\[ h = 0.1 \quad x(1) = 0.7502 \quad y(1) = -1.1894 \]
\[ h = 0.05 \quad x(1) = 0.7454 \quad y(1) = -1.0679 \]

difference: \(-0.0048\) \(-0.1215\)

The phase plane figure compares the results with the two step sizes for \( 0 \leq t \leq 1 \), starting at the initial condition (1,1).

The approximation using the larger step size is less accurate than that using the smaller step size; the exact solution of the IVP lies slightly inside the two curves shown.
3. \[ x' = y \quad x(0) = 1 \]
\[ y' = -x - y^3 \quad y(0) = 1 \]
We approximate the solution of the IVP on the interval \(0 \leq t \leq 1\) using Euler's method with step sizes \(h=0.1\) and \(h=0.05\). The results are summarized in the following table and figure.

<table>
<thead>
<tr>
<th>(h=0.1)</th>
<th>(h=0.05)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(t)</td>
<td>(x)</td>
</tr>
<tr>
<td>0</td>
<td>1.1</td>
</tr>
<tr>
<td>0.1</td>
<td>1.18</td>
</tr>
<tr>
<td>0.2</td>
<td>1.2439</td>
</tr>
<tr>
<td>0.3</td>
<td>1.2934</td>
</tr>
<tr>
<td>0.4</td>
<td>1.3292</td>
</tr>
<tr>
<td>0.5</td>
<td>1.3516</td>
</tr>
<tr>
<td>0.6</td>
<td>1.3606</td>
</tr>
<tr>
<td>0.7</td>
<td>1.3561</td>
</tr>
<tr>
<td>0.8</td>
<td>1.338</td>
</tr>
<tr>
<td>0.9</td>
<td>1.3064</td>
</tr>
<tr>
<td>1</td>
<td>1.3064</td>
</tr>
</tbody>
</table>

The Euler approximation using the smaller step size is consistently lower in \(x\) and higher in \(y\) than that using the larger step size. We expect the exact solution to follow this pattern.

4. \[ x' = y \quad x(0) = 1 \]
\[ y' = -x - y - y^3 \quad y(0) = 1 \]
We approximate the solution of the IVP on the interval \(0 \leq t \leq 1\) using Euler's method with step sizes \(h=0.1\) and \(h=0.05\). The results are summarized in the following table and figure.

<table>
<thead>
<tr>
<th>(h = 0.1)</th>
<th>(h = 0.05)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(t)</td>
<td>(x)</td>
</tr>
<tr>
<td>0</td>
<td>1.1</td>
</tr>
<tr>
<td>0.1</td>
<td>1.17</td>
</tr>
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<td>0.2</td>
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<td>0.3</td>
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</tr>
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<td>0.5</td>
<td>1.266</td>
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<td>1.2544</td>
</tr>
<tr>
<td>0.7</td>
<td>1.2314</td>
</tr>
<tr>
<td>0.8</td>
<td>1.1983</td>
</tr>
<tr>
<td>0.9</td>
<td>1.1565</td>
</tr>
<tr>
<td>1</td>
<td>1.1565</td>
</tr>
</tbody>
</table>

The figure for #4 is almost the same as the figure for #3, and the tables confirm that. However, we note that in #4 \(y\) changes slightly more quickly, and therefore so does \(x\).
### Changing Views

An *xy* phase portrait does not indicate the speed at which a point moves while tracing a trajectory. We can observe more effects of using Euler's method with different step sizes by looking at *tx* and *ty* graphs. In all cases a tighter curvature occurs with smaller step size.

A *scaled* vector field shows *speed*, helping to give a rough idea of how these three graphs relate. Note that in some cases we should have used larger bounds for *x* and *y*.

5. \[ x' = y^2 \quad x(0) = -1 \]
   \[ y' = x^2 \quad y(0) = -0.7 \]

For \(0 \leq t \leq 5\), the figures show the requested views and include the scaled vector field for a sample (nonsymmetric) initial condition. Over the given ranges the difference in approximation due to stepsize is hardly visible in these reduced images.

![](image1.png)

Euler approximations (with \(h = 0.1, 0.05\)) for \(x' = y^2, \ y' = x^2\)

(b) For this system we find \(\text{speed} = \sqrt{(x')^2 + (y')^2} = \sqrt{x^4 + y^2}\), which means trajectories rise ever more quickly in the *x* and *y* directions as their distance from the origin increases.

6. \[ x' = x + y \quad x(0) = 0 \]
   \[ y' = x + y \quad y(0) = 0.1 \]

For \(0 \leq t \leq 2\), the figures show the requested views and include the scaled vector field, for a sample IC. The phase portrait is the same for both step sizes; we cannot see that it gives larger *x* and *y* more quickly for smaller *h*.

![](image2.png)

Euler approximates (with \(h = 0.1, 0.05\)) for \(x' = x + y, \ y' = x + y\)

(b) For this system we find \(\text{speed} = \sqrt{(x')^2 + (y')^2} = \sqrt{2 |x+y|}\), which increases with distance from the line \(x+y=0\) and causes *tx* and *ty* curves to turn upward.
7. \[ \begin{align*}
x' &= y \\
y' &= -x
\end{align*} \]
\[ x(0) = 0 \]
\[ y(0) = 1 \]

For \( 0 \leq t \leq 2\pi \), the figures show the requested views and include the scaled vector field for a sample initial condition.

Euler approximations (with \( h = 0.1, 0.05 \)) for \( x' = y, \ y' = -x \)

(b) For this system we find speed = \( \sqrt{(x')^2 + (y')^2} = \sqrt{(y)^2 + (x)^2} \), equal to the distance from the origin. We note that for \( t = 2\pi \) the Euler approximations come full circle for \( x \)-values; for smaller step size the \( y \)-value of the end of the approximation is closer to the initial condition, which would be the end of the exact solution to the harmonic oscillator.

8. \[ \begin{align*}
x' &= y \\
y' &= -x + x^3
\end{align*} \]
\[ x(0) = -1.93 \]
\[ y(0) = 2.00 \]

For \( 0 \leq t \leq 2\pi \), the figures show the requested views and include the scaled vector field for a sample initial condition.

Euler approximations (with \( h = 0.1, 0.05 \)) for \( x' = y, \ y' = -x + x^3 \)

(b) For this system we find speed = \( \sqrt{(x')^2 + (y')^2} = \sqrt{y^2 + x^2(x^2 - 1)^2} \).
9. \( x' = y \quad x(0) = 0 \)
\( y' = -x - x^3 \quad y(0) = 1 \)

For \( 0 \leq t \leq 2\pi \), the figures show the requested views and include the scaled vector field.

(b) For this system we find speed \( = \sqrt{(x')^2 + (y')^2} = \sqrt{y^2 + x^2 + 2x^4 + x^6} \). Compare both the speed and the trajectories with those of Problem 7, which uses the same initial condition. Here the slopes become steeper as \( |x| \) increase (causing "cycles" to be vertically elongated) and faster as well (causing a cycle to be completed for \( t < 2\pi \)). Note that the smaller stepsize, the longer the period between "cycles". As in Problem 7, the exact solution would produce closed cycles.

10. \( x' = y \quad x(0) = -9 \)
\( y' = -\sin x \quad y(0) = 0 \)

For \( 0 \leq t \leq 6\pi \), the figures show the requested views and include the scaled vector field.

(b) For this system we find speed \( = \sqrt{(x')^2 + (y')^2} = \sqrt{y^2 + x^2} \), hence the trajectories move ever more quickly in the \( y \) direction as distance from the \( x \)-axis increases; the effect of \( \sin^2 x \) oscillates between 0 and 1 and has less effect for larger \( |y| \).

An exact solution to this IVP would cycle back to the IC; we notice that the Euler approximations escape the cycle, and have a longer "period" for smaller step size.

The equilibrium at \((-2\pi, 0)\) is between the saddles at \((\pi, 0)\) and \((-3\pi, 0)\). It must hence be a center because the vector field is circling but otherwise symmetric about the \( x \)-axis.
Changing Parameters

11. \[
x' = x + \varepsilon x(1-x^2-y^2)
\]
\[
y' = -x + \varepsilon y(1-x^2-y^2)
\]

The figures show phase portraits for \( \varepsilon = 3, 1, 0.1, 0, -0.1, -0.9, -1, -1.1, -3 \), with exaggerated equilibria dots.

As positive \( \varepsilon \) approaches zero, we observe an unstable node at the origin and two saddles fixed at (0,1) and (0,-1), while two stable nodes move further apart along the line \( y=-x \).

When \( \varepsilon = 0 \), the whole vertical axis becomes a line of unstable equilibria. As \( \varepsilon \) becomes negative, unstable nodes appear at (0,1) and (0,-1), and the origin becomes a saddle.

As \( \varepsilon \) reaches -1, the unstable nodes remain fixed at (0,1) and (0,-1), but the origin becomes a stable node and two saddles appear, moving apart along \( y=-x \) as \( \varepsilon \) becomes more negative.

Note the drastic differences in location and type of equilibria that occur between each pair of phase portraits as \( \varepsilon \) decreases.
12. \[ x' = y \]
\[ y' = -x + \epsilon \left(1 - x^2\right)y \]

The phase portraits shown illuminate the role of \( \epsilon \).

The only equilibrium is at \((0,0)\), for all values of \( \epsilon \). Critical values of \( \epsilon \) are \( \epsilon = 0 \), where the origin’s stability changes, and \( \epsilon = \pm 2 \), where the origin is between a node and a spiral.

For \( \epsilon < 0 \), the origin is stable, but we see in the phase portraits there is an unstable limit cycle that sends far away trajectories off toward infinity. At \( \epsilon = 0 \) (not shown) van der Pol’s equation is simply the harmonic oscillator, with circular clockwise trajectories centered at the origin.

For \( \epsilon > 0 \), the origin is unstable, but we find a stable limit cycle, drawing in trajectories from far away as well as from the origin. This case is of the most interest, because the long term behavior will be periodic. Note: As \( \epsilon \) increases, the limit cycles become increasingly irregular in shape.

When \( \epsilon > 1 \), the van der Pol system experiences "relaxation oscillations", where energy is slowly stored and then suddenly released almost instantaneously. Note in the phase portrait for \( \epsilon = 5 \) that trajectories head straight for the \( x \)-axis, only at the last moment they turn almost at right angles, indicating a sudden change of motion. The final figures shows \( x(t) \) and \( y(t) \) for \( \epsilon = 5 \), to show these relaxation oscillations. With an initial condition close to the origin, we note that the solution reaches periodicity almost instantaneously.

*Note the change of scale for \( \epsilon = 5 \). Axes now extend to 8 in all directions in order to catch the height of the limit cycle.
For Problems 13-15 we extended the simple spreadsheet with Euler's method used in Problems 1-4, by inserting columns for $z$ and $z\dot{}$. We graph $x$, $y$, and $z$ as functions of $t$, and summarize the numerical results in a table.

Where the graphs show very different results for step sizes $h = 0.05$ and 0.1 you know Euler is not giving a good approximation. If you needed numerical accuracy, you would want to use a more accurate method, such as Runge-Kutta (See Problems 16-22), which would require a more elaborate spreadsheet or DE software that can handle a system of three DEs. We have done this and provided the results from Runge-Kutta (RK) for comparison with Euler (E).

13.  
\[ \begin{align*} 
    x' &= x + y & x(0) &= 1 \\
    y' &= y + z & y(0) &= 1 \\
    z' &= -y + 2z & z(0) &= 1 
\end{align*} \]

<table>
<thead>
<tr>
<th>Method</th>
<th>$h$</th>
<th>$x(1)$</th>
<th>$y(1)$</th>
<th>$z(1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Euler</td>
<td>0.1</td>
<td>5.8751</td>
<td>4.6810</td>
<td>1.3996</td>
</tr>
<tr>
<td>Euler</td>
<td>0.05</td>
<td>6.2473</td>
<td>4.7888</td>
<td>1.1948</td>
</tr>
<tr>
<td>Runge-Kutta</td>
<td>0.1</td>
<td>6.6604</td>
<td>4.8746</td>
<td>0.9325</td>
</tr>
<tr>
<td>Runge-Kutta</td>
<td>0.05</td>
<td>6.6604</td>
<td>4.8746</td>
<td>0.9324</td>
</tr>
<tr>
<td>Exact*</td>
<td></td>
<td>6.6604</td>
<td>4.8746</td>
<td>0.9324</td>
</tr>
</tbody>
</table>

The graphs show that the numerical results give very close results until about $t = 0.3$; after that the method matters most for $z(t)$ and least for $y(t)$.

*Because this system is linear, it is also possible to obtain an exact formula solution as in Chapter 6, but it is messy to do by hand. We used Maple.
14. \[ x' = -x + xy \quad x(0) = 1 \]
\[ y' = y + xz \quad y(0) = 1 \]
\[ z' = -y + yz \quad z(0) = 2 \]

<table>
<thead>
<tr>
<th>Method</th>
<th>( h )</th>
<th>( x(1) )</th>
<th>( y(1) )</th>
<th>( z(1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Euler</td>
<td>0.1</td>
<td>8.3791</td>
<td>14.9571</td>
<td>19.4341</td>
</tr>
<tr>
<td>Euler</td>
<td>0.05</td>
<td>30.5143</td>
<td>48.6815</td>
<td>71.5855</td>
</tr>
<tr>
<td>Runge-Kutta</td>
<td>0.1</td>
<td>overflow</td>
<td>overflow</td>
<td></td>
</tr>
<tr>
<td>Runge-Kutta</td>
<td>0.05</td>
<td>overflow</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Comparison of two Euler approximations with Runge-Kutta

15. \[ x' = x + y \quad x(0) = 2 \]
\[ y' = -x + tz \quad y(0) = 1 \]
\[ z' = z + x^2 \quad z(0) = 1 \]

<table>
<thead>
<tr>
<th>Method</th>
<th>( h )</th>
<th>( x(1) )</th>
<th>( y(1) )</th>
<th>( z(1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Euler</td>
<td>0.1</td>
<td>5.9022</td>
<td>1.7499</td>
<td>20.8353</td>
</tr>
<tr>
<td>Euler</td>
<td>0.05</td>
<td>6.1817</td>
<td>2.4982</td>
<td>23.0535</td>
</tr>
<tr>
<td>Runge-Kutta</td>
<td>0.1</td>
<td>6.4202</td>
<td>2.9697</td>
<td>25.5269</td>
</tr>
<tr>
<td>Runge-Kutta</td>
<td>0.05</td>
<td>6.4993</td>
<td>3.2136</td>
<td>25.7431</td>
</tr>
</tbody>
</table>

Comparison of two Euler approximations with Runge-Kutta
### Epidemic

16. \( S' = -\alpha SI \quad S(0) = 950 \)
\( I' = \alpha IS - \beta I \quad I(0) = 30 \)
\( R' = \beta I \quad R(0) = 20 \) (From the fact that the total \( S + I + R = 1000 \))

We solve the IVP using Euler's method with stepsize \( h = 0.1 \) and obtain the following results as printed out by Maple. After 20 days the number of infected is \( I(20) = 10 \), the number susceptible is \( S(20) = 916 \), and the number of recovered is \( R(20) = 74 \).

<table>
<thead>
<tr>
<th>Day</th>
<th>I</th>
<th>S</th>
<th>R</th>
<th>Day</th>
<th>I</th>
<th>S</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>30.0000</td>
<td>950.0000</td>
<td>20.0000</td>
<td>11</td>
<td>16.1263</td>
<td>926.8134</td>
<td>57.0603</td>
</tr>
<tr>
<td>1</td>
<td>28.3866</td>
<td>947.2233</td>
<td>24.3901</td>
<td>12</td>
<td>15.2244</td>
<td>925.3578</td>
<td>59.4178</td>
</tr>
<tr>
<td>3</td>
<td>25.3952</td>
<td>942.1326</td>
<td>32.4722</td>
<td>14</td>
<td>13.5635</td>
<td>922.6927</td>
<td>63.7438</td>
</tr>
<tr>
<td>4</td>
<td>24.0110</td>
<td>939.8019</td>
<td>36.1871</td>
<td>15</td>
<td>12.7998</td>
<td>921.4740</td>
<td>65.7263</td>
</tr>
<tr>
<td>5</td>
<td>22.6970</td>
<td>937.6037</td>
<td>39.6992</td>
<td>16</td>
<td>12.0776</td>
<td>920.3254</td>
<td>67.5970</td>
</tr>
<tr>
<td>6</td>
<td>21.4503</td>
<td>935.5308</td>
<td>43.0188</td>
<td>17</td>
<td>11.3949</td>
<td>919.2430</td>
<td>69.3621</td>
</tr>
<tr>
<td>7</td>
<td>20.2680</td>
<td>933.5762</td>
<td>46.1558</td>
<td>18</td>
<td>10.7497</td>
<td>918.2230</td>
<td>71.0273</td>
</tr>
<tr>
<td>8</td>
<td>19.1472</td>
<td>931.7332</td>
<td>49.1196</td>
<td>19</td>
<td>10.1400</td>
<td>917.2618</td>
<td>72.5982</td>
</tr>
<tr>
<td>9</td>
<td>18.0851</td>
<td>929.9956</td>
<td>51.9193</td>
<td>20</td>
<td>9.5639</td>
<td>916.3562</td>
<td>74.0799</td>
</tr>
<tr>
<td>10</td>
<td>17.0790</td>
<td>928.3576</td>
<td>54.5634</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

See the following figure for plots of these values.
Epidemic Formula

17. Given

\[
\frac{dl}{dS} = \frac{\alpha IS - \beta I}{-\alpha IS} = \frac{\alpha S - \beta S}{-\alpha S} = -1 + \frac{\beta}{\alpha}
\]

Integrating yields the general solution

\[I(S) = -S + \frac{\beta}{\alpha} \ln I(S) + c\]

where \(c\) is an arbitrary constant. Substituting the initial condition \(I(S_0) = I_0\), yields

\[I_0 = -S_0 + \frac{\beta}{\alpha} \ln(S_0) + c\]

Solving for \(c\) yields

\[c = I_0 + S_0 - \frac{\beta}{\alpha} \ln(S_0)\]

Hence, a relationship between \(I\) and \(S\).

\[I(s) = -S + \frac{\beta}{\alpha} \ln(S) + I_0 + S_0 - \frac{\beta}{\alpha} \ln(S_0)\]

\[= I_0 + S_0 - S + \frac{\beta}{\alpha} \ln \left( \frac{S}{S_0} \right)\]

This equation yield \(I\) as a function of \(S\), which is shown in the figure for \(\alpha = 0.00025\), \(\beta = 0.2\) and \(I_0 = 30\), \(S_0 = 930\).

A typical experiment is illustrated by \(SI\), \(tl\), and \(tS\) graphs for \(S_0 = 950; I_0 = 40, 30, 20\).
# Bug Race

18. We solve the three IVPs in the phase plane, with \( x(0)=0, y(0)=1 \).

\begin{align*}
\text{A:} & \quad x' = y \\
& \quad y' = -x \\
\text{B:} & \quad x' = y \\
& \quad y' = -x + x^3 \\
\text{C:} & \quad x' = y \\
& \quad y' = -x - x^3
\end{align*}

Bug A returns to the starting point when \( t = 2\pi \), because we know that the solution of that system is \( x(t) = \sin t \).

From the phase portrait, we see that Bug B does not have a chance; her DE takes her to infinity. Bug C however has an inside track, always advancing faster up or down than A. Bug C returns to the starting point first, therefore C wins the race.

## Runge-Kutta Method

In Problems 19-24, we compare the numerical approximation of \( x(t) \) using Euler's method (from Problems 1-6) and the Runge-Kutta method, each at step sizes \( h = 0.05, 0.1 \). The tables show that the Runge-Kutta method refines the approximation beyond (but in the same direction) as using Euler's method with a smaller step size. Note also that in most cases Runge-Kutta does such a good job that its approximation does not change appreciably with step size, and is very close to the exact solution.

We graph \( x(t) \) for both methods, using \( h = 0.1 \). Several graphs are extended past \( t = 1 \) until we can predict long-term behavior, to show how sometimes approximations alternately diverge and converge.

19. \begin{align*}
x' &= y \\
y' &= -x + x^3 - y
d \begin{array}{|c|c|c|} \hline 
\text{Method} & h & x(1) & y(1) \\
\hline 
\text{Euler} & 0.1 & 1.9596 & 1.6559 \\
\text{Euler} & 0.05 & 2.0126 & 1.8071 \\
\text{Runge-Kutta} & 0.1 & 2.0823 & 2.0095 \\
\text{Runge-Kutta} & 0.05 & 2.0823 & 2.0095 \\
\hline 
\end{array}
\end{align*}

The \( xy \) graph in Problem 1 confirms the steady rise of \( x \) values as \( t \) increases. Here we see that solutions quickly go off screen, and a glance at the equations shows that as \( x \) is increasingly greater than 1, \( y' \) grows ever bigger, which increases \( x' \) and increases \( x \), faster and faster.
20. \[ x' = y \quad x(0) = 1 \]
\[ y' = -x - x^3 - y \quad y(0) = 1 \]

<table>
<thead>
<tr>
<th>Method</th>
<th>( h )</th>
<th>( x(1) )</th>
<th>( y(1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Euler</td>
<td>0.1</td>
<td>0.7502</td>
<td>-1.1894</td>
</tr>
<tr>
<td>Euler</td>
<td>0.05</td>
<td>0.7454</td>
<td>-1.0679</td>
</tr>
<tr>
<td>Runge-Kutta</td>
<td>0.1</td>
<td>0.7451</td>
<td>-0.9649</td>
</tr>
<tr>
<td>Runge-Kutta</td>
<td>0.05</td>
<td>0.7451</td>
<td>-0.9649</td>
</tr>
</tbody>
</table>

The \( xy \) graph in Problem 2 confirms that for \( t \) between 0 and 1, \( x(t) \) rises and then falls as \( t \) increases. Extending the \( t \) axis shows that \( x(t) \) will exhibit damped oscillation, which is appropriate because the system is close to a damped harmonic oscillator for small \( x \). When \( x < 1 \), \( x^3 \) contributes little to \( y' \); once \( y \) gets close to zero, \( y \) also contributes little to \( y' \).

Here we can see that the Runge-Kutta method gives higher values than Euler for both \( x(1) \) and \( y(1) \), but the approximations merge near \( t = 2, 4, \) and \( 7 \), only to diverge again (by ever-smaller amounts) between these values. We also observe that although the approximations are close for \( x(1) \), they were further apart near \( t = 0.5 \).

21. \[ x' = y \quad x(0) = 1 \]
\[ y' = -x - y^3 \quad y(0) = 1 \]

<table>
<thead>
<tr>
<th>Method</th>
<th>( h )</th>
<th>( x(1) )</th>
<th>( y(1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Euler</td>
<td>0.1</td>
<td>1.3064</td>
<td>-0.4467</td>
</tr>
<tr>
<td>Euler</td>
<td>0.05</td>
<td>1.2864</td>
<td>-0.4197</td>
</tr>
<tr>
<td>Runge-Kutta</td>
<td>0.1</td>
<td>1.2667</td>
<td>-0.3939</td>
</tr>
<tr>
<td>Runge-Kutta</td>
<td>0.05</td>
<td>1.2667</td>
<td>-0.3939</td>
</tr>
</tbody>
</table>

The arguments given for Problems 20 and 2 apply, but we observe that here the maximal separations become more pronounced as \( t \) becomes larger, and that here both approximations tend to cross the \( t \)-axis at the same time. Because these equations are even closer to a damped harmonic oscillator for small \( y \) than those of Problem 17 these observations seem reasonable.
22. \[ x' = y \quad x(0) = 1 \]
\[ y' = -x - y - y^3 \quad y(0) = 1 \]

<table>
<thead>
<tr>
<th>Method</th>
<th>( h )</th>
<th>( x(1) )</th>
<th>( y(1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Euler</td>
<td>0.1</td>
<td>1.1565</td>
<td>-0.4885</td>
</tr>
<tr>
<td>Euler</td>
<td>0.05</td>
<td>1.1484</td>
<td>-0.4525</td>
</tr>
<tr>
<td>Runge-Kutta</td>
<td>0.1</td>
<td>1.1397</td>
<td>-0.4188</td>
</tr>
<tr>
<td>Runge-Kutta</td>
<td>0.05</td>
<td>1.1397</td>
<td>-0.4188</td>
</tr>
</tbody>
</table>

The arguments given for Problems 20 and 2 apply, though the separations are less pronounced, and the axis crossings are further apart.

23. \[ x' = y^2 \quad x(0) = 1 \]
\[ y' = x^2 \quad y(0) = 1 \]

<table>
<thead>
<tr>
<th>Method</th>
<th>( h )</th>
<th>( x(1) )</th>
<th>( y(1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Euler</td>
<td>0.1</td>
<td>6.1289</td>
<td>6.1289</td>
</tr>
<tr>
<td>Euler</td>
<td>0.05</td>
<td>9.5527</td>
<td>9.5527</td>
</tr>
<tr>
<td>Runge-Kutta</td>
<td>0.1</td>
<td>81.9964</td>
<td>81.9964</td>
</tr>
<tr>
<td>Runge-Kutta</td>
<td>0.05</td>
<td>163.9834</td>
<td>163.9834</td>
</tr>
</tbody>
</table>

Both table and graph emphasize that Runge-Kutta approximations shoot up much sooner than Euler approximations when slopes are becoming steeper. In Problem 5 we chose as initial condition, \((-1, -0.7)\), to be nonsymmetric and more general than \((1,1)\). But as you can predict from the DEs, the Runge-Kutta approximations will show similar dramatic steepness.
24. \( x' = x + y \quad x(0) = 1 \)
\( y' = x + y \quad y(0) = 1 \)

<table>
<thead>
<tr>
<th>Method</th>
<th>( h )</th>
<th>( x(1) )</th>
<th>( y(1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Euler</td>
<td>0.1</td>
<td>6.1917</td>
<td>6.1917</td>
</tr>
<tr>
<td>Euler</td>
<td>0.05</td>
<td>6.7275</td>
<td>6.7275</td>
</tr>
<tr>
<td>Runge-Kutta</td>
<td>0.1</td>
<td>7.3889</td>
<td>7.3889</td>
</tr>
<tr>
<td>Runge-Kutta</td>
<td>0.05</td>
<td>7.3890</td>
<td>7.3890</td>
</tr>
</tbody>
</table>

Both table and graph emphasize that Runge-Kutta approximations shoot up sooner than Euler approximations when slopes are becoming steeper (though the equations indicate the effects will be less dramatic than those in Problem 23). In Problem 6 we chose as initial condition, \((0, 0.1)\), to be nonsymmetric and more general than \((1,1)\). But as you can predict from the DEs, the Runge-Kutta approximations will show similar steeper slopes than Euler.

■ Proper Step Size

25. \( x' = y \quad x(0) = 1 \)
\( y' = -x \quad y(0) = 0 \)

The table lists various approximations for the solution at \( t = 2\pi \); the best of each method is in bold type.

<table>
<thead>
<tr>
<th>Method</th>
<th>( h )</th>
<th>( x(2\pi) )</th>
<th>( y(2\pi) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Euler</td>
<td>0.1</td>
<td>1.365735</td>
<td>0.0283219</td>
</tr>
<tr>
<td>Euler</td>
<td>0.05</td>
<td>1.169518</td>
<td>0.0060963</td>
</tr>
<tr>
<td>Euler</td>
<td>0.01</td>
<td>1.031902</td>
<td>0.0002159</td>
</tr>
<tr>
<td>Euler</td>
<td>0.005</td>
<td>1.015828</td>
<td>0.0000053</td>
</tr>
<tr>
<td>Euler</td>
<td>0.001</td>
<td>1.003146</td>
<td>-0.0000026</td>
</tr>
<tr>
<td>Runge-Kutta</td>
<td>0.1</td>
<td>0.99999957</td>
<td>0.00000049</td>
</tr>
<tr>
<td>Runge-Kutta</td>
<td>1.0</td>
<td>0.963439</td>
<td>0.03227</td>
</tr>
</tbody>
</table>

Coarsest approximations, on the outside by Euler \((h = 0.1)\), on the inside by Runge-Kutta \((h = 1.0)\).

The middle solution, by Runge-Kutta \((h = 0.1)\), is very close to the circle of the exact solution.

Note, from \( x(2\pi) \) and from the graph that Euler approximations always return \textit{outside} the unit circle and Runge-Kutta approximations always return \textit{inside} the unit circle.

Note also that the latter returns closer to \((1, 0)\) with \( h = 0.1 \) than Euler does with \( h = 0.001 \).
Additional Bifurcations

26. \[
\begin{bmatrix}
  x' \\
  y'
\end{bmatrix}
= \begin{bmatrix}
  -1 & 1 \\
  \alpha & \alpha
\end{bmatrix}
\begin{bmatrix}
  x \\
  y
\end{bmatrix}
\]
The characteristic equation
\[
p(\lambda) = \begin{bmatrix}
  -1 - \lambda \\
  \alpha & \alpha - \lambda
\end{bmatrix}
= \lambda^2 + (1 - \alpha)\lambda - 2\alpha = 0
\]
has roots
\[
\lambda_1, \lambda_2 = -\frac{1}{2} + \frac{1}{2}\alpha \pm \frac{1}{2}\sqrt{1 + 6\alpha + \alpha^2}.
\]

(a) For \(\alpha = -2\) the eigenvalues are \(-\frac{3}{2} \pm \frac{\sqrt{7}}{2}\), so the origin is a stable spiral equilibrium.

(b) At \(\alpha = -0.1\) Example 3 shows a stable equilibrium node, so there must be a bifurcation value between \(-2\) and \(-0.1\) that separates spiral and node behaviors. The bifurcation we seek occurs when the eigenvalues change from real to complex, i.e., when \(\alpha^2 + 6\alpha + 1 = 0\). This condition is satisfied for
\[
\alpha_2, \alpha_3 = -\frac{6 \pm \sqrt{36 - 4}}{2} \approx -0.17157288, -5.828425.
\]
Hence, we find two bifurcation values where equilibria change between spiral and node behaviors. The bifurcation point between \(-0.1\) and \(-2\) is at \(\alpha_2 \approx -0.171575\).

(c) We found in part (b) that the bifurcation for \(\alpha < -2\) occurs at \(\alpha_3 \approx -5.828425\).

When a bifurcation occurs between a node and a spiral, the Jacobian changes from having two distinct real eigenvalues and two distinct eigenvectors to a double eigenvalue with a single eigenvector (at bifurcation) to complex conjugate eigenvalues with no real eigenvectors. See part (d) for pictures.

(d) The changing eigenvector situation can be seen in the phase portraits. When there are two eigenvectors, there are two directions to draw straight lines through the origin along the direction field; at bifurcation there is only one direction in which that can be done; with a spiral equilibrium point there is no such direction. The following figures illustrate this, first about \(\alpha = -5.828425\), then (on the next page) about \(\alpha = -0.17157288\).
SECTION 7.3     Numerical Solutions     737

$\alpha = -2$

spiral, no eigenvectors

$\alpha = -0.17157288$

one eigenvector

$\alpha = -0.1$

two eigenvectors

- **Hopf Bifurcation**
  
  27. Student Lab Project with IDE

- **Saddle Node Bifurcation**
  
  28. Student Lab Projects with IDE

- **Suggested Journal Entry**
  
  29. Student Project
7.4 Chaos, Strange Attractors & Period Doubling

Equilibrium Analysis

1. \( \dot{x} = f(x, y, z) = 10(y - x) \)
   \( \dot{y} = g(x, y, z) = 28x - y - xz \)
   \( \dot{z} = h(x, y, z) = xy - \frac{8}{3}z \)

The equilibrium points are found by solving
\[
10(y - x) = 0 \\
28x - y - xz = 0 \\
xy - \frac{8}{3}z = 0
\]
yielding three points: \((0, 0, 0)\), \((6\sqrt{2}, 6\sqrt{2}, 27)\), and \((-6\sqrt{2}, -6\sqrt{2}, 27)\).

- Near \((0, 0)\) the linearized equations are

\[
\begin{align*}
\dot{x} &= -10x + 10y \\
\dot{y} &= 28x - y \\
\dot{z} &= -\frac{8}{3}z
\end{align*}
\]
with Jacobian matrix
\[
J = \begin{bmatrix}
-10 & 10 & 0 \\
28 & -1 & 0 \\
0 & 0 & -\frac{8}{3}
\end{bmatrix}
\]
and three real eigenvalues: \(-\frac{8}{3}, -5.5 \pm \frac{1}{2}\sqrt{1201} \approx -2.67, 11.83, -22.83\).

Because one eigenvalue is positive, the equilibrium point \((0, 0)\) is unstable.

- Near the other equilibria, we set \(u = x - x_0\), \(v = y - y_0\), \(w = z - z_0\); then

\[
\begin{bmatrix}
\dot{u} \\
\dot{v} \\
\dot{w}
\end{bmatrix}
= \begin{bmatrix}
f_x & f_y & f_z \\
g_x & g_y & g_z \\
h_x & h_y & h_z
\end{bmatrix}
\begin{bmatrix}
u \\
v \\
w
\end{bmatrix}
= \begin{bmatrix}
-10 & 10 & 0 \\
28 - z & -1 & -x \\
y & x & -\frac{8}{3}
\end{bmatrix}
\begin{bmatrix}
u \\
v \\
w
\end{bmatrix}
\]

\[
J(6\sqrt{2}, 6\sqrt{2}, 27) = \begin{bmatrix}
-10 & 10 & 0 \\
1 & -1 & -6\sqrt{2} \\
6\sqrt{2} & 6\sqrt{2} & -\frac{8}{3}
\end{bmatrix}
\]
\[
J(-6\sqrt{2}, -6\sqrt{2}, 27) = \begin{bmatrix}
-10 & 10 & 0 \\
1 & -1 & 6\sqrt{2} \\
-6\sqrt{2} & -6\sqrt{2} & -\frac{8}{3}
\end{bmatrix}
\]

Eigenvalue calculation (which can be done quickly using appropriate software such as Maple or Matlab) gives the same values for both matrices: \(\lambda_1 = -13.8\), \(\lambda_2, \lambda_3 = 0.09 \pm 0.2i\).

Hence both nonzero equilibrium points are unstable (due to the positive real part of \(\lambda_2, \lambda_3\)).

In summary, we have shown that all the equilibria of the Lorenz system with \(r = 28\) are unstable, which helps explain why an orbit can never settle and the system could exhibit a strange attractor.
Hypersensitivity

2. Figure 7.4.2 in the text shows a typical example of all possible graphs for the Lorenz equations (see Problem 1), with two sets of initial conditions. For both sets of IC, each time series \((tx, ty, tz)\) starts out the same, but soon becomes quite different; changes in the other views are more subtle. Here we compare behaviors for three different values of \(x(0)\), with fixed \(y(0)\) and \(z(0)\).

To estimate \(t_{div}\) when solutions visibly diverge, we need a time series graph; the figures show \(tx\) graphs, vertically aligned to allow easiest estimates of \(t_{div}\), as well as \(xyz\) graphs.

(a) \(x(0) = -6.0, y(0) = 12, z(0) = 12\)

A sensitive system needs a very good approximation method; all figures were calculated by the Runge-Kutta method, with step size \(h = 0.05\) and \(0 \leq t \leq 10\).

(b) \(x(0) = -6.01, y(0) = 12, z(0) = 12\)

Compared with (a), \(t_{div} \approx 9.3\).

(c) \(x(0) = -6.1, y(0) = 12, z(0) = 12\)

Compared with (a), \(t_{div} \approx 4.3\);

With a much larger difference in IC, divergence happens much sooner than in part (b).

Note: Different solvers also show sensitivity to initial conditions, so your graphs may differ somewhat even for these initial conditions. The final figure shows an example, with a different solver but over the same range of \(0 \leq t \leq 10\), for \((x_0, y_0, z_0) = (-6.12, 12)\). Comparison with the first figure of part (a) shows \(t_{div} \approx 7\).

CAUTION: To study sensitivity to IC only, you must use the same solver, as well as the same method and stepsize. See Problem 8 for further exploration of sensitivity issues.
Long-Term Behavior

3. For the Lorenz equations (Problem 1) with two sets of initial conditions as in Problem 2, we let the time series of $x(t)$ run to $t = 100$. Here we show the $ty$ and $tz$ graphs as well, and observe that all the time series show divergence before $t = 10$. On the other hand, with longer time series the $xyz$ phase portraits show fewer gaps and tend to look more similar than in Problem 2.

(a) $x(0) = -6.0, y(0) = 12, z(0) = 12$

(b) $x(0) = -6.01, y(0) = 12, z(0) = 12$
4. Using the Lorenz equations (Problem 1) with IC $x(0) = 0$, $y(0) = 1$, $z(0) = 25$, the graphical solution of the Lorenz equation is shown in the $yz$-plane for $0 \leq t \leq 100$.

The rough left/right symmetry we see in the figure is explained by the product $xy$ in $dz/dt$:

- $-x, -y$ produces the same $xy$ as $x, y$;
- $-x, y$ produces the same $xy$ as $x, -y$.

The $yz$ view lets us look down the $z$ axis and see both positive and negative $x$ and $y$ values projected onto the $yz$ plane.

5. (a) We linearize the Lorenz equations (Problem 1) near zero by simply dropping off the nonlinear terms, yielding

$$\begin{align*}
\dot{x} &= -10x + 10y \\
\dot{y} &= 28x - y \\
\dot{z} &= -\frac{8}{3}z
\end{align*}$$

or, in matrix form,

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -8/3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

(b) For $0 \leq t \leq 1$ the figures show two views, $yz$ and $tx$, of the linearized equations with $x(0) = -6$, $y(0) = 12$, $z(0) = 12$, in comparison with the same views for the nonlinear equations over the same time interval (shortened because the linearized curves go off screen quickly).

(c) There is only one equilibrium for the linearized Lorenz system, at $(0, 0, 0)$.

The eigenvalues (see Problem 1) are approximately $-2.67$, $11.83$, $-22.83$: two negative but one positive. Hence the origin is unstable. Solutions to the linearized system will simply go off to infinity in the unstable direction, so chaos cannot result.
Roessler Views

6. \[ \dot{x} = f(x, y, z) = -y - z \]
\[ \dot{y} = g(x, y, z) = x + 0.2y \]
\[ \dot{z} = h(x, y, z) = -5.7z + xz + 0.2 \]

For good approximations in a sensitive chaotic system we use the Runge-Kutta method with \( h = 0.05 \) to create the phase plane and time series graphs in the figures.

(a) The three different 2-D phase plane projections of the Roessler system are shown below, with IC \((0,0,0)\) and \(0 \leq t \leq 100\). The \(xy\) trajectory is attracted to a somewhat periodic looping. The \(xz\) and \(yz\) trajectories show that \(z\) never goes negative, at least for this IC.

(b) Because there is cyclic motion in the \(xy\)-plane, we expect motion in the \(tx\)- and \(ty\)-planes to oscillate; in fact the time series shown, for \(0 \leq t \leq 100\), appear to move into an approximate 3-cycle after \(t = 50\). We also observe that \(z(t)\) peaks more randomly, spends a lot of time at \(z = 0\), and never goes negative, at least for this IC \((0,0,0)\).

CAUTION: We must not assume too much from these graphs, which seem to settle into a pattern. If we let \(t\) run on to 250, we see some surprises (below). Between \(t=150\) and \(t=200\) we see a burst of chaotic motion (which maybe settles down later to a triple cycle pattern like that observed above, but now we should have some doubts. Further exploration and analysis is in order. Such “intermittency”, alternating chaos with approximate order, can indeed occur in nonlinear dynamics. Beware of jumping to conclusions too soon!

Extended time series, to \(t = 250\)
Roessler Analysis

7. (a) To find the equilibrium points of the Roessler equations we solve

\[
\begin{align*}
    x' &= f(x, y, z) = -y - z = 0 \\
    y' &= g(x, y, z) = x + 0.2y = 0 \\
    z' &= h(x, y, z) = -5.7z + xz + 0.2 = 0
\end{align*}
\]

yielding the two equilibrium points at approximately

\[ (0.0070, -0.0351, 0.0351) \text{ and } (5.69297, -28.4648, 28.4648) \]

Linearizing the system around each point yields

\[
\begin{bmatrix}
    u' \\
    \dot{v} \\
    \ddot{w}
\end{bmatrix} =
\begin{bmatrix}
    f_x & f_y & f_z \\
    g_x & g_y & g_z \\
    h_x & h_y & h_z
\end{bmatrix}
\begin{bmatrix}
    u \\
    v \\
    w
\end{bmatrix} =
\begin{bmatrix}
    0 & -1 & -1 \\
    1 & 0.2 & 0 \\
    0 & 0 & -5.7 + x
\end{bmatrix}
\begin{bmatrix}
    u \\
    v \\
    w
\end{bmatrix},
\]

where we evaluate the Jacobian matrix at the individual equilibrium points.

- Near \((0.007, -0.035, 0.035)\) we get

\[
J = \begin{bmatrix}
    0 & -1 & -1 \\
    1 & 0.2 & 0 \\
    0.035 & 0 & -5.693
\end{bmatrix},
\]

with eigenvalues \(\lambda_1 = -5.69\), \(\lambda_2\), \(\lambda_3 = 0.097 \pm i\).

Because the complex eigenvalues have positive real parts, the system is unstable. However, the real parts are almost zero, so the stability is almost borderline. Note that the real eigenvalue is negative, so solutions will approach a plane.

- Near \((5.7, -28.5, 28.5)\) we get

\[
J = \begin{bmatrix}
    0 & -1 & -1 \\
    1 & 0.2 & 0 \\
    28.5 & 0 & -0.007
\end{bmatrix},
\]

with eigenvalues \(\lambda_1 = 0.2\), \(\lambda_2\), \(\lambda_3 = 0.0034 \pm 5.4i\).

In this case, the real eigenvalues are positive and the complex ones have positive real parts so the equilibrium point is unstable in all directions. The real part of the complex roots is almost zero, however, so it is close to crossing the borderline for stability.

(b) As a sample stability study, we compare \(tx\) graphs for two close sets of initial conditions. The figures show that soon after \(t = 30\) these solutions become dramatically different.

\[
\begin{align*}
    \text{IC } x(0) &= -6.0, \\
    y(0) &= 12, \ z(0) = 12
\end{align*}
\]

\[
\begin{align*}
    \text{IC } x(0) &= -6.1, \\
    y(0) &= 12, \ z(0) = 12
\end{align*}
\]
Sensitivity

8. \[ \begin{align*}
\dot{x} &= -y - z \\
\dot{y} &= x + 0.2y \\
\dot{z} &= -5.7z + xz + 0.2
\end{align*} \]

\( x(0) = 1 \) \( y(0) = 0 \) \( z(0) = 0 \)

To make comparisons for sensitive systems:
- vary only one aspect at a time;
- superimpose graphs (with different colors) or align and highlight comparisons with ruler lines.

(a) **Sensitivity to numerical method:** the following graphs show a great sensitivity in the numerical solution \( x, y \) to the approximate method used for the same initial condition and stepsize. The equations are so sensitive to small differences in \( x, y, z \) that after a short time the solutions show no resemblance to each other.

The figures show the nonstability of Roessler’s equations between

- *Euler’s method* (heavy curve);
- *Runge-Kutta* (lighter curve).

Fixed stepsize \( h = 0.05 \)

(b) **Sensitivity to stepsize:**

*Euler* approximations of Roessler’s equations show nonstability with stepsizes \( h = 0.1 \) (heavy curve), \( h = 0.05 \) (lighter curve).

Note that it takes little time before \( tx \) approximations diverge.

Euler approximations \((tx, 0 \leq t \leq 50)\)

On the other hand the *Runge-Kutta* method is not as sensitive to stepsize. For \( h = 0.1 \) (heavy curve), \( h = 0.05 \) (lighter curve),

Runge-Kutta approximations (see figure) are so close together that the difference is much smaller (barely distinguishable for much of the graph; we note also that oscillations are smaller than with Euler approximations above, because Runge-Kutta more closely approximates exact solutions.

Runge-Kutta approximations \((tx, 0 \leq t \leq 50)\)

(c) **Sensitivity to solver:** See Problem 2; the final note that gives a last figure to compare with the first figure is a good example. You’ll find many more whenever you try to reproduce someone else’s pictures of a sensitive or chaotic system.

(d) We have seen how numerical approximations can differ greatly for small differences in initial conditions, stepsizes, methods, and computer numerics (Problems 2-3, 7 & 8). These differences will be most obvious in time series \((tx, ty, tz)\) pictures; strange attractors in phase space pictures for a long run of \( t \) will look much more similar. When you notice differences that start small and get bigger, you can try to minimize them by using smaller stepsizes and more robust approximation methods (such as Runge-Kutta).
**Varying Lorenz**

9. Student Lab Project with IDE

**Bifurcations in Lorenz System**

10. Student Lab Project with IDE

**Varying Roessler**

11. $x' = -y - z$
    $y' = x + 0.2 y$
    $z' = -rz + xz + 0.2$

This system is highly sensitive, so we make figures using the Runge-Kutta method with stepsize $h \leq 0.05$. To see phase plane cycles, we must eliminate transient solutions, so the trajectories are plotted only for $300 \leq t \leq 500$.

(a) We show typical $xy$ graphs that bracket bifurcation values between 1-, 2-, and 4-cycles.

We found intermediate values in each left-right pair where we could first observe the period doubling; then we turned to $tx$ graphs to fine tune our estimates.

A time series must extend over many cycles to confirm that a cycle persists (see Problem 7(b) re intermittency), as well as to distinguish a truly doubled cycle period from one that simply converges very slowly to the previous cycle period. In fact, a *long* time series shrunk to a short picture width can help highlight the difference.

single peak/cycle, $r = 2.8$

double peak/cycle, $r = 2.9$

$0 \leq t \leq 1000$; levels marked at left side
(b) We found an 8-cycle at $r = 4.2$, an emerging 16-cycle at $r = 4.22$.

![8-cycle](image)

Emerging 16-cycle

![Emerging 16-cycle](image)

(c) Continuing to increase past $r = 5$ we found additional pairs that bracketed bifurcations.

The suggested sudden change from Chaos to 3-cycle occurs somewhere with $5.19 \geq r \leq 5.20$.

![Chaotic Orbit](image)

Sudden 3-cycle

![Sudden 3-cycle](image)

The 3-cycle persists for a surprisingly long span of $t$, then finally doubles to a 6-cycle somewhere with $5.38 \leq r \leq 5.39$.

![3-cycle](image)

6-cycle

![6-cycle](image)
12. (a) The direction field for \( y' = y(1 - y) \) is shown in the figure. We see that \( y = 1 \) is a stable equilibrium point and \( y = 0 \) is an unstable point. Solutions starting at positive \( y(0) \neq 1 \) approach 1.

NOTE: \( t = nh \)

(b) We show Euler’s method with various stepsizes \( h \).

When \( h = 1.8 \) Euler’s method reaches steady state of 1.000000 after 54 iterations using Microsoft Excel and 6-place accuracy. The figure shows ever-smaller oscillations around 1 just before reaching 1.000000.

1-cycle: 1.000 \( h = 1.8 \)

When \( h = 2.2 \) Euler’s approximation becomes periodic with repeating values 1.162844, 0.746247.

2-cycle: 11.162844, 0.746247 \( h = 2.2 \)

When \( h = 2.5 \) we have period doubling where now the numbers repeat in blocks of four.

4-cycle: 1.1578, 0.7012, 2.5 1.2250, 0.5359 \( h = 2.5 \)

When \( h = 2.55 \) we have another period doubling with the numbers repeating in blocks of eight (to six-place accuracy after 114 iterates).

8-cycle: 11.1313, 0.7524, 1.2274, 2.55 0.5156, 1.1524, 0.7045, 1.2354, 0.4939 \( h = 2.55 \)

When \( h = 2.57 \) there is no periodicity (to six places) of any period in the numbers in the table, although an extended graph shows the sequence is nearly periodic.

\( n \) from 1500 to 2000 \( h = 2.57 \)

When \( h = 2.6 \) the number show no periodicity of any kind to six places. The extended graph shows repeated highs and lows but with no sustained regularity.

\( n \) from 1500 to 2000 \( h = 2.6 \)
(c) Experimenting with the spreadsheet program Microsoft Excel, we let $h$ increase in value as we look for the $h$-values where the behavior changes qualitatively, we used $y(0) = 0.5$.

At $h = 1$ we observe the first bifurcation:

- When $h \leq 1$, Euler approximations approach 1 from below;
- When $h > 1$, Euler approximations oscillate about 1 before settling at 1.000000

At $h = 2$ there is another qualitative change in behavior:

- When $h \leq 2$, Euler approximations eventually reach 1.000000.
- When $h > 2$, the approximations settle into a 2-cycle oscillating above and below 1.

The next bifurcation value occurs between $h = 2.44$ and $h = 2.45$:

- When $h = 2.44$ the steady state oscillates between the two values $1.196253$ and $0.6233242$
- When $h = 2.45$ we get the repeated four-cycles of (roughly) $1.193238, 0.628326, 1.200481, \text{and } 0.610832$.

We let you narrow down the subsequent bifurcations by experimenting on a spreadsheet.

The value of $h$ where four-cycles end and eight-cycles begin is between 2.5 and 2.55.

After that the period-doubling points are closer and closer together, and it will be harder to zero in on those points. You may require more significant places on $h$ to find them; try 10-significant-place accuracy in the spreadsheet.

(d) A strange thing happens for $h > 3$. The Euler approximation jumps around chaotically and eventually goes to minus infinity. The larger the value of $h$, the sooner it goes to minus infinity. As shown, when $h = 3.04$ it goes to minus infinity around $t = 50$, but when $h = 3.01$, it doesn’t go to minus infinity until around $t = 150$.

![Graph](image)

Suggested Journal Entry

13. Student Project
7.5 Chaos in Forced Nonlinear Systems

Problems 1–8 (and later Problem 15) are highly exploratory experiments such as are needed in research. We compute these curves numerically with a graphic DE solver (usually by Runge-Kutta with small step size), because an analytic solution of nonlinear DEs is (at best) awkward and difficult to obtain. Organization becomes critical in order to report results. Our solutions show some possible ways to do this. Students own experiments may lead to other ways. The goal in every case is concise communication.

Damped Pendulum

1. \( \ddot{\theta} + 0.1\dot{\theta} + \sin \theta = F \cos \omega t \) or, as a system, \( \begin{align*} \dot{\theta} &= y, \\ \dot{y} &= F \cos \omega t - \sin \theta - 0.1y \end{align*} \)

(a) The unforced equation (with \( F = 0 \)) has equilibria when \( \dot{\theta} = 0 \) and \( \sin \theta = 0 \), i.e., at \( \theta = \pm n\pi, \quad n = 0, 1, 2, \ldots \).

Sample trajectories in the \( \theta\dot{\theta} \) plane show that \((0, 0), (\pm2\pi, 0), (\pm4\pi, 0), \ldots \) appear to be stable spiral points; \((\pm\pi, 0), (\pm3\pi, 0), \ldots \) appear to be unstable saddle points.

A trajectory starting at almost any point will come to rest at one of the stable equilibria.

The single trajectory starting at \((3, 1)\), shown below, spirals toward \((2\pi, 0)\); the time series of \(\theta\) and \(\dot{\theta}\) are consistent with the trajectory in the \(\theta\dot{\theta}\) plane.

(b) The forced equation (with \( F = 1, \quad \omega = 1 \)) shows (next page) quite a different set of figures: the trajectory from \((3, 1)\) approaches periodic motion around \((2\pi, 0)\); the time series of \(\theta\) and \(\dot{\theta}\) settle into periodic cycles.

(c) The forcing term \( \cos t \) can be interpreted as a sinusoidal pushing of a damped pendulum. Comparing figures (a) and (b) shows that starting at \((3, 1)\), the unforced pendulum comes to rest at \((2\pi, 0)\) while the forced pendulum oscillates around \((2\pi, 0)\). These results agree with intuition on the behavior of a damped pendulum.
2. The center of the trajectories in Problem 1 is at the point \((2\pi, 0)\).

If we linearize the pendulum equation \(\ddot{\theta} + 0.1\dot{\theta} + \sin \theta = F \cos \omega t\)
about that point, we get \(\ddot{\theta} + 0.1\dot{\theta} + (\theta - 2\pi) = F \cos \omega t\).

(a) The unforced linearized equation (with \(F = 0\)) starting at \((3,1)\) follows an inward spiral
towards the stable equilibrium point \((2\pi, 0)\), but as the following figures show, the
\(\dot{\theta}\)-axis must be slightly extended to \([-4, 4]\) to accommodate the linearized trajectory.

(b) The forced linearized equation (with \(F = 1, \omega = 1\)) causes the motion starting at \((3,1)\) to
approach periodic behavior around \((2\pi, 0)\), as is expected with periodic forcing of a
pendulum. However, we note a dramatic difference in \(\theta\) and \(\dot{\theta}\) scales (see figures) from
the unforced case. The forced trajectory spirals outward, to a cycle of radius \(\text{ten}!\)

(c) We note that in both (a) and (b) linearization requires significant changes of scale from
the nonlinear system, especially in the case of forcing. Furthermore, we observe that for
the unforced pendulum the equilibrium is a \textit{sink}, while for the forced pendulum the
equilibrium is a \textit{source}, with trajectories approaching a periodic solution (of surprisingly
large radius in the linearized case).
Solution of the Linearized Pendulum

3. If we linearize the pendulum equation 
 \[ \ddot{\theta} + 0.1\dot{\theta} + \sin \theta = F \cos \omega t \]
about \((2\pi,0)\), we get 
\[ \ddot{\theta} + 0.1\dot{\theta} + (\theta - 2\pi) = F \cos \omega t \]

(a) The unforced linearized equation 
\[ \ddot{\theta} + 0.1\dot{\theta} + \theta = 2\pi \]
has the homogeneous solution
\[ \theta_h(t) = e^{-0.05t} (c_1 \cos t + c_2 \sin t), \]
and a particular solution can easily be found to be \( \theta_p(t) = 2\pi \), so the general solution is
\[ \theta(t) = e^{-0.05t} (c_1 \cos t + c_2 \sin t) + 2\pi. \]
Substituting the IC \( \theta(0) = 3 \) and \( \dot{\theta}(0) = 1 \) yields
\[ c_1 = 3 - 2\pi \approx -3.28 \]
\[ c_2 = 1 + 0.05c_1 \approx 0.836. \]
Hence, the linearized unforced pendulum IVP solution is
\[ \theta(t) = -e^{-0.05t} (3.28 \cos t - 0.836 \sin t) + 2\pi, \]
which approaches the steady state of \( 2\pi \), agreeing with the \( t\theta \) figure of Problem 2(a).

(b) For the forced linearized pendulum equation 
\[ \ddot{\theta} + 0.1\dot{\theta} + \theta = \cos t + 2\pi \]
we use the method of undetermined coefficients to find \( \theta_p(t) \); then the general solution is
\[ \theta(t) = e^{-0.05t} (c_1 \cos t + c_2 \sin t) + 2\pi + 10 \sin t. \]
Substituting the IC \( \theta(0) = 3 \) and \( \dot{\theta}(0) = 1 \) yields
\[ c_1 = 3 - 2\pi \approx -3.28 \]
\[ c_2 = -9 + 0.05c_1 \approx -9.16. \]
Hence, the linearized forced pendulum IVP solution is
\[ \theta(t) = -e^{-0.05t} (3.28 \cos t + 9.16 \sin t) + 2\pi + 10 \sin t. \]
The first term of this algebraic solution gives a sinusoidal oscillation that disappears as \( t \to 100 \); we can see its diminishing effect in the \( t\theta \) figure of Problem 2(b). The remaining two terms indicate the eventual steady state sinusoidal oscillation about \( 2\pi \), and also confirms the surprisingly large amplitude of 10 that we see in Problem 2(b).
Nonlinear versus Linear Pendulums

4. Using the results from Problems 1–3, we summarize the differences between the nonlinear and linearized pendulum equations, both forced and unforced, in the following figure table. The phase plane graphs have been redrawn to the same scale to allow better comparison, all trajectories start at \((3, 1)\).

\[
\begin{align*}
\text{Nonlinear} & \\
\ddot{\theta} + 0.1\dot{\theta} + \sin \theta = F \cos \omega t \\
\text{Unforced, Nonlinear} & \\
\text{Forced, Nonlinear} & \\
\text{Linear, about } 2\pi & \\
\ddot{\theta} + 0.1\dot{\theta} + (\theta - 2\pi) = F \cos \omega t \\
\text{Unforced, linearized} & \\
\text{Forced, Linearized} &
\end{align*}
\]

Same-scale comparison of the pendulum equation: linearized vs. nonlinear, forced vs. unforced.

We observe that for the IC \(\theta(0) = 3\) and \(\dot{\theta}(0) = 1\) all systems cycle about \((2\pi, 0)\).

Both unforced systems spiral in toward \((2\pi, 0)\).

Both forced systems approach a steady state periodic motion about \((2\pi, 0)\), although not the same periodic motion – shapes and sizes are very different.
Chaos Exploration of a Forced Damped Pendulum

5. (a) We examine motion under \( \ddot{\theta} + 0.1 \dot{\theta} + \sin \theta = F \cos \omega t \) in three different cases:

- If the pendulum starts at \((3, 1)\), it settles down to periodic motion about \((2\pi, 0)\).
- If the pendulum starts at \((-1, 1)\), it settles directly into periodic motion about \((0, 0)\). But most trajectories are not so simple:
- If the pendulum starts at \((-1, -1)\), it behaves much more chaotically before settling down to periodic motion about yet another equilibrium, at \((-2\pi, 0)\).

Forced damped pendulum, phase plane graphs for three different IC

The following graphs of \( \theta \) versus \( t \) show these differences by the height of the cycles. Each move between levels of \( 2\pi \) represents the pendulum going over the top in one direction or the other.

Each time series shown is for the phase plane trajectory shown directly above.

Forced damped pendulum \( \theta(t) \) graphs for different IC

(b) The eleven initial conditions suggested will give \( \theta(t) \) graphs that settle out at several different levels, representing different numbers of swings “over the top.” The figure shows \( \theta(t) \) graphs for the first five ICs. There appears to be no pattern that would allow prediction of level for a given initial condition, and different computer software will give results that differ in detail.

Note: The vertical \( \theta \) scale runs from \(-10\) to \(15\), much extended from the graphs in Part (a).
Period Doubling; Poincaré Sections in Forced Damped Pendulum

6. Student project using IDE software.

Double-Well Potential

7. \( \ddot{x} + bx - x + x^3 = F \cos \omega t \), or, as a system, \( \dot{x} = y \)
\[ \dot{y} = F \cos \omega t - by + x - x^3 \]

(a) The damped, unforced equation \( \ddot{x} + bx - x + x^3 = 0 \) has equilibria where \( \dot{x} = 0 \) and at the same time \( x \) is a root of \( x - x^3 = 0 \), i.e., when \( (x, \dot{x}) = (0,0), (\pm 1,0) \). It has been shown in the text Example 1 of Section 7.2 that \( (0,0) \) is an unstable saddle and that \( (\pm 1,0) \) are stable spiral points. Except for the separatrices of the saddle, trajectories spiral towards one of the stable equilibria; those that pass close to the saddle must veer away.

(b) The damped forced equation \( \ddot{x} + \dot{x} - x + x^3 = \cos t \) is not autonomous, so trajectories can cross themselves. The figure shows a single trajectory, from \((-1,1)\), that is messy, but appears to settle down into a nice triple loop (similar to that seen for the Lorenz attractor in Fig. 7.4.6), which is clear if the initial transient part (dotted) of the trajectory is removed.

The solution is not chaotic if it settles down to a pattern that persists (don’t jump to conclusions too soon – see 7.4 Problem 6, final set of figures). Further investigation is provided in the next Problem (8) of the current section.

(c) The undamped unforced equation \( \ddot{x} - x + x^3 = 0 \) has two equilibria \((-1,0)\) and \((1,0)\) that are center points, whereas in the damped case they were asymptotically stable spiral points. The origin is still a saddle case as before, so except for the separatrices of the saddles, the solutions are periodic.

(d) The undamped but forced equation \( \ddot{x} - x + x^3 = \cos t \) is not autonomous, so phase plane trajectories can cross themselves. The single trajectory from \((-1,1)\) in the figure appears to do so chaotically, similar to text Figure 8.5.6(c). Without the damping shown in part (b), this behavior does not settle into periodic motion.
Forced Duffing Oscillator, A Route to Chaos

8. \( \ddot{x} + 0.25 \dot{x} - x + x^3 = F \cos t \) or, as a system, \( \dot{x} = y \)
\[ \dot{y} = F \cos t - 0.25y + x - x^3 \]

Problem 7(b) is a special case; here we explore some of the more hidden complexities.

(a) To determine the motion starting near the origin, we set \( F = 0 \) and start at four points: \((0, \pm 0.25)\) and \((\pm 0.25, 0)\).

Two solutions approach the stable equilibrium point \((1, 0)\); two solutions approach \((-1, 0)\), as shown.

The 45-degree line \( \dot{x} = -x \) separates the two basins of attraction; trajectories from initial points above the line converge to \((1, 0)\); those that start below the line converge to \((-1, 0)\).

An interesting (and difficult) problem would be to determine the boundary of the basin of attraction analytically from the DE, without the phase portrait.

(b) As a sample experiment, we chose the initial point \((0.25, 0)\) and applied forcing terms \(F \cos t\) with amplitudes \(F = 0.1, 0.2, 0.25\); the phase portraits are shown in the figure. Notice the increasingly chaotic (unpredictable) behavior as the amplitude increases. If you allowed the trajectories to run forever, for the larger \(F\) values, some limit cycles may appear around one of the original equilibria at \((\pm 1, 0)\). The decision of which of these points to circle around seems very sensitive to changes in either \(F\) or the initial conditions.

Period-Doubling Exploration, Forced Duffing Oscillator

9. Student IDE Project
Chemical Oscillators

10. \[ x' = 0.4 + x^2 y - 2.2 x + 0.05 \cos \omega t \]
    \[ y' = 1.2 x - x^2 y \]

This is a sensitive system, so for the figures we used Runge-Kutta with \( h = 0.05 \). To see the cycles and test their persistence, the graphs run from \( t = 200 \) to \( t = 500 \).

\( \omega = 0.6 \) 1-cycle

For a time series we chose \( t \times \) because \( x \) is the direction of widest spread.

\( \omega = 0.75 \) 2-cycle

The little bumps are just for the dimples in the loop; they do not represent separate cycles.

\( \omega = 0.78 \) 4-cycle

We also could see an 8-cycle at \( \omega = 0.785 \), but the differences are too small to see at this reduced scale.

\( \omega = 0.8 \) chaotic trajectory

Even when we let \( t \) run to 2000, new loops were still being drawn.

*As a final note we observe that these pictures demonstrate unequivocally that doubling refers to the \( n \)-cycle period at the bifurcation \( r \)-value, not to the elapsed time for a \( 2n \)-cycle at any subsequent \( r \)-value.*
11. \[ \dot{x} = -xy^2 + 0.999 \cos \omega t \]

\[ \dot{y} = xy^2 - y \]

For this sensitive system we used Runge-Kutta with \( h = \frac{1}{128} \) (step sizes that are powers of 2 give faster computer calculations). In order to bypass the transient solution and check for persistence of cycles, graphs are made with \( 500 \leq t \leq 800 \).

(a) We found a period-doubling sequence as shown:

\[ \omega = 1.85, \text{ 1-cycle} \]

\[ \omega = 1.8, \text{ 2-cycle} \]

\[ \omega = 1.77, \text{ 4-cycle} \]

We also found at \( \omega = 1.76 \) an 8-cycle, but the split is too small to show up well on the scale of our pictures.

A big surprise, however, was that for \( \omega = 1.75 \) we did not get a chaotic trajectory, but rather a clean 6-cycle. We suspect that our numerical approximation technique was finer, and/or our \( t \) run was longer, than the problem-writer expected. In fact it took many experiments to discover that the bifurcation value lies extremely close to 1.75 (between 1.7505 and 1.7502), so it would be easy to miss.

(b) Quasiperiodicity means a solution returns at regular intervals close to, but not exactly on, the previous ‘cycle’. Because the equation is nonautonomous, we find the phase plane graph (left column, next page) is too complicated to unravel without a time series (right column, next page) to help explain it. The \( tx \) graphs show how the various relative maximums and minimums keep changing.

Experimentation with higher \( t \) values would make a good group project; you should have enough information to start forming questions to guide further exploration.

*Pictures displayed on next page.*
Note: in the following figures the transients have been eliminated from the $xy$-phase portrait, but they remain on the $tx$-time series.

$\omega = 3$
(not requested but easier to read!)

$\omega = 3.5$

$\omega = 4$

$\omega = 4.5$
Forced van der Pol Equation

12. $\ddot{x} - \varepsilon(1-x^2)\dot{x} + x = F \cos \omega t$

(a) $\varepsilon = 0.1$, $F = 0.5$, $\omega = 1$

The solution quickly reaches a steady state that looks pretty close to a pure harmonic oscillation between 3.2 and –3.2.

(b) $\varepsilon = 1$, $F = 0.5$, $\omega = 1$

The solution quickly reaches a steady state resembling a rather warped sine or cosine curve.

(c) $\varepsilon = 1$, $F = 1$, $\omega = 0.3$

The solution gives rise to a more complicated periodic motion, with three peaks per period are displayed.

These figures start at $t = 50$ to eliminate the transient solution and isolate the

(d) $\varepsilon = 1$, $F = 1$, $\omega = 0.4$

Here a different periodic solution is observed.

These figures start at $t = 50$ to eliminate the transient solution and isolate the cycle.
Poincaré Sections for Periodic Functions

13. (a) \( x(t) = \sin t \), period \( \pi \), starting at \( t = 0 \). We compute \( \dot{x}(t) = \cos t \) and

\[
\begin{align*}
x(0) &= \sin 0 = 0 & \dot{x}(0) &= \cos 0 = 1 \\
x(\pi) &= \sin \pi = 0 & \dot{x}(\pi) &= \cos \pi = -1 \\
x(2\pi) &= \sin 2\pi = 0 & \dot{x}(2\pi) &= \cos 2\pi = 1 \\
&\quad \ldots \\
x(n\pi) &= \sin n\pi = 0 & \dot{x}(n\pi) &= \cos n\pi = (-1)^n.
\end{align*}
\]

Hence, the Poincaré section consists of the two points \((x, \dot{x}) = (0, 1), (0, -1)\).

Starting at a different \( t_0 \) would give the two points \((x(t_0), \dot{x}(t_0)) = (a, b)\) and \((-a, -b)\).

If the period were a multiple of \(2\pi\), then the Poincaré section would give a single point.

(b) \( x(t) = \sin t \), period \( \frac{\pi}{4} \), starting at \( t = 0 \). We compute \( \dot{x}(t) = \cos t \) and

\[
\begin{align*}
x(0) &= \sin 0 = 0 & \dot{x}(0) &= \cos 0 = 1 \\
x\left(\frac{\pi}{4}\right) &= \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2} & \dot{x}\left(\frac{\pi}{4}\right) &= \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2} \\
x\left(\frac{\pi}{2}\right) &= \sin \frac{\pi}{2} = 1 & \dot{x}\left(\frac{\pi}{2}\right) &= \cos \frac{\pi}{2} = 0 \\
&\quad \ldots \\
\end{align*}
\]

Continuing in this manner, we obtain a cycle of eight points on the unit circle in the \(x\dot{x}\) plane, at the point \((0, 1)\) and moving around the circle in jumps of 45 degrees.

Starting at a different \( t_0 \) would lead to another eight point cycle as follows: the evenly spaced eight point cycle starting at \( t = 0 \) would be rotated through the angle \( t_0 \) subends.

If the period were a multiple of \(2\pi\), then the Poincaré section would give a single point.

(c) \( x(t) = \sin 2t \), period \( \pi \), starting at \( t = 0 \). We compute \( \dot{x}(t) = 2\cos 2t \) and

\[
\begin{align*}
x(0) &= \sin 0 = 0 & \dot{x}(0) &= 2\cos 0 = 2 \\
x(\pi) &= \sin 2\pi = 0 & \dot{x}(\pi) &= 2\cos 2\pi = 2 \\
x(2\pi) &= \sin 4\pi = 0 & \dot{x}(2\pi) &= 2\cos 4\pi = 2 \\
&\quad \ldots \\
\end{align*}
\]

Continuing in this manner gives the single point \((0, 2)\).

Starting at a different \( t_0 \) would give the single point \((x(t_0), \dot{x}(t_0)) = (a, b)\).

If the period were a multiple of \(2\pi\), the results would be the same as for period \(\pi\).

\textit{Continued on next page}
(d) \( x(t) = \sin 2t + \sin t \), period \( \pi \), starting at \( t = 0 \). We compute \( \dot{x}(t) = 2\cos 2t + \cos t \)

\[
\begin{align*}
  x(0) &= \sin 0 + \sin 0 = 0 & \dot{x}(0) &= 2\cos 0 + \cos 0 = 3 \\
  x(\pi) &= \sin 2\pi + \sin \pi = 0 & \dot{x}(\pi) &= 2\cos 2\pi + \cos \pi = 1 \\
  x(2\pi) &= \sin 4\pi + \sin 2\pi = 0 & \dot{x}(2\pi) &= 2\cos 4\pi + \cos 2\pi = 3 \\
  x(3\pi) &= \sin 6\pi + \sin 3\pi = 0 & \dot{x}(3\pi) &= 2\cos 6\pi + \cos 3\pi = 1 \\
  x(4\pi) &= \sin 8\pi + \sin 4\pi = 0 & \dot{x}(4\pi) &= 2\cos 8\pi + \cos 4\pi = 3 \\
  \cdots & & \cdots
\end{align*}
\]

From the preceding computations, we conclude that the Poincaré section consists of the 2 points \((0, 3)\) and \((0, 1)\).

Starting at a different \( t_0 \) would give a messier orbit.

If the period were \( 2\pi \), the Poincaré section would be a single point, \((0, 3)\).

14. For this sum of two noncommensurate periodic functions, we have

\[
\begin{align*}
  x(t) &= \sin(2\pi t) + \sin t \\
  \dot{x}(t) &= 2\pi\cos(2\pi t) + \cos t.
\end{align*}
\]

We first find the \( x\dot{x} \) Poincaré section by starting at \( t = 0 \) and strobing at \( t = 0, 2\pi, 4\pi, \ldots \). Doing this yields the points shown in the table.

<table>
<thead>
<tr>
<th>point</th>
<th>( t )</th>
<th>( x )</th>
<th>( \dot{x} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0.000</td>
<td>2( \pi + 1 ) = 7.56</td>
</tr>
<tr>
<td>2</td>
<td>2( \pi )</td>
<td>0.978</td>
<td>-0.301</td>
</tr>
<tr>
<td>3</td>
<td>4( \pi )</td>
<td>-0.405</td>
<td>-4.746</td>
</tr>
<tr>
<td>4</td>
<td>6( \pi )</td>
<td>-0.811</td>
<td>4.679</td>
</tr>
<tr>
<td>5</td>
<td>8( \pi )</td>
<td>0.7406</td>
<td>5.222</td>
</tr>
<tr>
<td>6</td>
<td>10( \pi )</td>
<td>0.504</td>
<td>-4.427</td>
</tr>
<tr>
<td>7</td>
<td>12( \pi )</td>
<td>-0.949</td>
<td>-0.975</td>
</tr>
<tr>
<td>8</td>
<td>14( \pi )</td>
<td>-0.111</td>
<td>7.244</td>
</tr>
<tr>
<td>\cdots</td>
<td>\cdots</td>
<td>\cdots</td>
<td>\cdots</td>
</tr>
<tr>
<td>( n+1 )</td>
<td>2( n\pi )</td>
<td>( \sin(4\pi^2 n) + \sin(2n\pi) )</td>
<td>( 2\pi\cos(4\pi^2 n) + \cos(2n\pi) )</td>
</tr>
</tbody>
</table>

We note that the 8th point is close but not exactly on the first point; this overlapping of subsequent points continues, producing a closed curve if the process continues forever.

The first eight points, and the closed curve formed by many more iterates, are shown in the figures on the next page.
Points 1 and 8 do not coincide. Point 1 is at \((0, 7.56)\).

Point 8 is at \((-0.1, 7.24)\) a little lower and left.

The periods of \(\sin 2\pi t\) and \(\sin t\) are not \textit{commensurate} (one not a rational number times the other). This type of function is called \textit{quasiperiodic} and can be seen by observing that the Poincaré section appears to lie along a smooth curve in the \(x\dot{x}\) plane.

To get a better idea of what is going on with this function, we plot the 1000 points of the Poincaré section for \(t = 0, \ 0.01\pi, \ 0.02\pi, \ldots \ 10\pi\); i.e., with a period one two-hundredth of the previous example.

We now draw the \textit{continuous} phase portrait behaviour of the curve \((x, \dot{x})\) for

\[ x(t) = \sin(2\pi t) + \sin t. \]

Note that the actual \((x(t), \dot{x}(t))\) curve winds around in the \(xx'\) plane in an almost periodic fashion. If the curve were strobed every \(2\pi\) units of time you would observe the simpler Poincaré section shown in the first two figures.
Stagecoach Wheels and the Poincaré Section

For Problems 15-18 the diagrams are a schematic sketch of the strobed motion, so that you can clearly see at which angle and when the strobe will light successive points.

15.

(a) $\omega_s = \frac{1}{2} \omega_0$.
Clock appears motionless.

(b) $\omega_s = \frac{1}{4} \omega_0$.
Clock appears motionless.

(c) $\omega_s = 2 \omega_0$.
Clock appears to alternate between the bottom and top positions.

(d) $\omega_s = 4 \omega_0$.
Clock appears to move clockwise in 15-second jumps.

(e) $\omega_s = \frac{2}{3} \omega_0$.
Clock appears to alternate between the top and bottom positions, with 1.5 actual rotations between strobe flashes.

(f) $\omega_s = \frac{3}{4} \omega_0$.
Clock appears to rotate clockwise in 20-second jumps, although it is actually moving clockwise in 80-second jumps.
16. \( \omega_s = \frac{q}{p} \omega_0 \). Here the hand travels the reciprocal \( \frac{p}{q} \) of the way around (clockwise) for every strobe, so after \( q \) strobes the hand is back at the start. Note that for \( \omega_s = \frac{q}{p} \omega_0 = 2 \omega_0 \)

then \( \frac{p}{q} = \frac{1}{2} \) which means the hand moves forward 30 seconds each strobe, as in Problem 13 (c).

17. (a) \( \omega_s = \frac{5}{2} \omega_0 \). The hand travels \( \frac{p}{q} = \frac{2}{5} \) of the way around (clockwise) for every strobe. After 5 strobes and 2 cycles of the clock, the hand is back at the start. The strobe flashes at

\[
0, \frac{2}{5}, \frac{4}{5}, \frac{6}{5}, \frac{8}{5}, \frac{10}{5} = 2,
\]

cauising apparent clockwise motion in 24-second jumps.

(b) \( \omega_s = \frac{5}{3} \omega_0 \). The hand travels \( \frac{p}{q} = \frac{3}{5} \) of the way around (clockwise) for every strobe. After 5 strobes and 3 cycles of the clock, the hand is back at the start. The strobe flashes at

\[
0, \frac{3}{5}, \frac{6}{5}, \frac{9}{5}, \frac{12}{5}, \frac{15}{5} = 3,
\]

cauising apparent clockwise motion in 36-second jumps.

(c) \( \omega_s = \frac{8}{3} \omega_0 \). The hand travels \( \frac{p}{q} = \frac{3}{8} \) of the way around (clockwise) for every strobe. After 8 strobes and 3 cycles of the clock, the hand is back at the start. The strobe flashes at

\[
0, \frac{3}{8}, \frac{6}{8}, \frac{9}{8}, \ldots, 8 \times \frac{3}{8} = 3,
\]

cauising apparent clockwise motion 22.5 seconds with each strobe flash; this makes the apparent motion rather erratic.

(Continued on next page.)
Continued from previous page.

(d) \( \omega_s = \frac{12}{11} \omega_b \). The hand travels \( \frac{p}{q} = \frac{11}{12} \) of the way around (clockwise) for every strobe. After 12 strobes and 11 cycles of the clock, the hand is back at the start. The strobe flashes at

\[
0, \frac{11}{12}, \frac{22}{12}, \frac{33}{12}, \ldots, 12 \times \frac{11}{12} = 11,
\]

cauising apparent counterclockwise motion in 5-second jumps.

(e) \( \omega_s = \frac{100}{101} \omega_b \). The hand travels \( \frac{p}{q} = \frac{101}{100} \) of the way around (clockwise) for every strobe. After 100 strobes and 101 cycles of the clock, the hand is back at the start. The strobe flashes at

\[
0, \frac{101}{100}, \frac{2 \times 101}{100}, \ldots, 100 \times \frac{101}{100} = 101,
\]

cauising apparent motion to be very slowly clockwise, in \(0.01 \times 60 = 0.6\) second jumps.

18. (a) The strobed motion has period 2 and moves in 30 second jumps. These means that \( \omega_s = 2 \omega_b \) or

\[
\omega_b = \frac{1}{2} \omega_s.
\]

(b) The strobed motion has period 4 and moves in 45 second jumps. These means that \( \omega_s = \frac{4}{3} \omega_b \) or \( \omega_b = \frac{3}{4} \omega_s \).
(c) The strobed motion has period 12 and moves in 25 second jumps. These means that $\omega_s = \frac{12}{5} \omega_0$ or

$$\omega_0 = \frac{5}{12} \omega_s.$$  

■ Suggested Journal Entry

19. Student Project